

# *Functions Whose Fourier Series Converge Uniformly for Every Change of Variable*

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1. Let  $f$  be a continuous function of period  $2\pi$ . We will say that  $f \in GW$  if the Fourier series of  $f \circ g$  converges everywhere for every homeomorphism  $g$  of  $[-\pi, \pi]$  with itself, and that  $f \in UGW$  if the series of  $f \circ g$  always converges uniformly. Goffman and Waterman [4] determined a necessary and sufficient condition for a function to belong to  $GW$ . In this note we give a necessary and sufficient condition for a function to belong to  $UGW$ . We also give an example of a function which is in  $GW$  but not in  $UGW$ .

Since bounded variation is conserved under change of variable, it follows from Jordan's theorem that  $UGW$  must contain all continuous functions of bounded variation. More generally, it must contain every continuous function of  $\Phi$ -bounded variation in the sense of L. C. Young for which the complementary function  $\Psi$  satisfies the Salem condition  $\sum \Psi(i^{-1}) < \infty$  [5], as well as the continuous functions with logarithmically integrable Banach indicatrix considered by Garsia and Sawyer [3]. Recently Waterman [6] introduced a change of variable invariant Banach space, the functions of "harmonic bounded variation," which contains all these  $\Phi$ -BV classes, but whose continuous members still have uniformly convergent Fourier series. In the other direction, when  $\sum \Psi(i^{-1}) = \infty$ , Baernstein [1] has constructed a continuous function of  $\Phi$ -bounded variation whose Fourier series diverges at a point. Thus  $\Phi$ -BV  $\subset UGW$  if and only if  $\sum \Psi(i^{-1}) = \infty$ , and also  $HBV \subset UGW$ .

2. As in [4], our condition may be phrased in terms of "systems of intervals" at a point. For each  $n$ , let  $I_{nm}$ ,  $m = 1, 2, \dots, k_n$ , be disjoint closed intervals such that  $I_{n,m-1}$  is to the left of  $I_{nm}$ . Let there be a real  $x$  such that for every  $\delta > 0$  there is an  $N$  such that  $I_{nm} \subset (x, x + \delta)$  whenever  $n > N$ . Then the collection  $\mathcal{I} = \{I_{nm}, n = 1, 2, \dots; m = 1, 2, \dots, k_n\}$  is called a right system of intervals at  $x$ . A left system is defined similarly. We remark that Goffman and Waterman assume in their definition of "system" that  $\lim_{n \rightarrow \infty} k_n = \infty$  and  $\lim_{n \rightarrow \infty} n^{-1}k_n = 0$ . In fact, the validity of (1) below for all systems is in-