

Lie Subalgebras of Finite Codimension in $\Gamma(TX)$

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The study of the infinite dimensional Lie group structure of $\text{Diff}^\infty(X)$, where X is a compact C^∞ -manifold was begun in [3]. The Lie algebra of this Lie group is $\Gamma^\infty(TX)$ and the investigation of $\text{Diff}^\infty(X)$ requires the study of this infinite dimensional Lie algebra. The new interest is with the infinite dimensional subalgebras. In this paper we show that every subalgebra of finite codimension which is closed in the C^∞ -topology is contained in the finite intersection of Lie subalgebras, each of which is the set of all vector fields vanishing at some fixed point.

In developing these ideas it is advantageous to drop the requirement that X be compact but work with $\Gamma_c^\infty(TX)$, the Lie algebra of smooth vector fields of compact support. X will always denote a 2nd countable, T_2 , C^∞ -manifold. Such an X is metrizable and we choose a metric, d , for X .

We first establish some local results. N will denote the set of nonnegative integers, and if $p = (p_1, \dots, p_n)$ is in N^n we define $|p| = p_1 + p_2 + \dots + p_n$. D_p will denote the partial derivative operator $\partial^{|p|}/\partial x_1^{p_1} \partial x_2^{p_2} \dots \partial x_n^{p_n}$. If V is an open subset of R^n , $\mathfrak{D}^\infty(V, R^m)$ will denote the C^∞ -mappings of V into R^m which have compact support. When $n = m$ this is a Lie algebra with $[\xi, \eta](x) = \xi'(x)\eta(x) - \eta'(x)\xi(x)$. Here $\xi'(x)$ denotes the derivative at x of the mapping $\xi: V \rightarrow R^n$, considered as a linear transformation of R^n into R^n ; similarly for η' .¹ $\pi_j: R^m \rightarrow R$ will denote the j^{th} projection so that we get $\mathfrak{D}^\infty(V, R^m) = \sum_1^m \mathfrak{D}^\infty(V, R)$ by $f = (f_1, f_2, \dots, f_m)$ where $f_i = \pi_i f$.

Lemma 1. (E. Borel). Suppose that $(c_p)_{p \in N^n}$ is a sequence of numbers indexed by N^n and that $x \in V$. Then there is an $f \in \mathfrak{D}^\infty(V, R)$ so that $(D_p f)(x) = c_p$ for all p in N^n .

Lemma 2. For $1 \leq k$, $1 \leq n$ let $S_{nk} = \{(r_1, r_2, \dots, r_n) \mid \text{each } r_i \text{ is in } N \text{ and } r_1 + r_2 + \dots + r_n = k\}$. Suppose that $L: S_{nk} \rightarrow R$ and that (x_1, \dots, x_n) is an n -tuple of numbers which satisfy: (s_1, \dots, s_n) in S_{nk+1} implies $\sum \{L(s_1, \dots,$