

Essential Self Adjointness for the Dirac Operator and Its Square

JOSEPH A. WOLF

Communicated by S. S. CHERN

§0. Introduction. Some of the natural operators in differential geometry are the Hodge–Laplace–Beltrami operator Δ on differential forms, the Kodaira–Hodge–Laplace operator \square on differential forms with values in a holomorphic vector bundle, and the Dirac operator D on spinors with values in certain vector bundles. The latter two are quite versatile because one can vary the bundle. In each case the operator is symmetric (formally self adjoint) and elliptic. If the base manifold is compact, then the operator has well defined spectrum (*i.e.*, is essentially self adjoint), which is discrete with finite multiplicities. The study of those spectra is one of the developing chapters in modern geometry.

When the base manifold is noncompact these operators need not be essentially self adjoint. However, they turn out to be essentially self adjoint, *i.e.*, to have well defined spectra, when the riemannian or hermitian metric on the base manifold is complete. Gaffney ([3], [4], [5]) proved Δ essentially self adjoint under conditions that he showed to be automatic for complete riemannian manifolds. Andreotti and Vesentini [1, §6] obtained estimates that show \square essentially self adjoint over a complete hermitian manifold. Parthasarathy [7] stated that one of the Andreotti–Vesentini estimates (see below) could be proved for D^2 by a similar method. Here we prove that D and D^2 are essentially self adjoint over a complete riemannian spin manifold. This has applications (*cf.* §7–10 below) to the problem of explicit realization of unitary representations of Lie groups, and we work in the generality suitable for construction of unitary representations.

In §1 we recall the Clifford algebra construction for spin groups and spin representations.

In §2 we establish the setting for our Dirac operators. Let Y be an oriented riemannian n -manifold and $\alpha: K \rightarrow SO(n)$ a Lie group homomorphism that factors through $\text{Spin}(n)$. The setting consists of a principal K -bundle $\mathfrak{F}_K \rightarrow Y$ and a connection Γ_K on \mathfrak{F}_K , such that α sends \mathfrak{F}_K to the oriented orthonormal frame bundle of Y and sends Γ_K to the riemannian connection. The case $\alpha: K \cong$