

On the Subharmonicity and Plurisubharmonicity of Geodesically Convex Functions

R. E. GREENE & H. WU

Communicated by S. S. CHERN

Introduction. A function $f: \mathbf{R}^n \rightarrow \mathbf{R}$ is said to be *convex* if its restriction to every straight line segment is a convex function in the one-variable sense, *i.e.*, if for any two vectors $V_1, V_2 \in \mathbf{R}^n$ and any real number $\lambda \in [0, 1]$,

$$f(\lambda V_1 + (1 - \lambda)V_2) \leq \lambda f(V_1) + (1 - \lambda)f(V_2).$$

Any C^2 convex function on \mathbf{R}^n is subharmonic because its second directional derivatives are all non-negative, *i.e.*, $V(Vf) \geq 0$ for any vector field V which is a linear combination with constant coefficients of the coordinate vector fields. An arbitrary convex function on \mathbf{R}^n is also necessarily subharmonic because it can be approximated uniformly on compact subsets of \mathbf{R}^n by C^∞ convex functions using the standard convolution smoothing process (*cf.* [5; p. 45]). Similarly, a function $f: \mathbf{C}^n \rightarrow \mathbf{R}$ which is convex when considered as a function from \mathbf{R}^{2n} to \mathbf{R} is plurisubharmonic because it is the limit uniformly on compact sets of a family of functions which are C^∞ and convex and which are therefore plurisubharmonic as a calculation with second directional derivatives shows. The purpose of this paper is to extend these results to geodesically convex functions on Riemannian manifolds in the case of the subharmonicity statement and to geodesically convex functions on Kähler manifolds in the case of the plurisubharmonicity statement. Precisely, a function $f: M \rightarrow \mathbf{R}$ on a C^∞ Riemannian manifold is said to be *geodesically convex* if its restriction to every geodesic segment is convex, *i.e.*, if for every geodesic $C(t)$ parametrized by arc length and defined for all $t \in [t_1, t_2]$

$$f(C(\lambda t_1 + (1 - \lambda)t_2)) \leq \lambda f(C(t_1)) + (1 - \lambda)f(C(t_2))$$

for all $\lambda \in [0, 1]$.

Theorem 1 asserts that a geodesically convex function on a C^∞ Riemannian manifold is subharmonic. Theorem 3 asserts that a geodesically convex function on a Kähler manifold is plurisubharmonic. As in the case of euclidean spaces, both these results are for C^2 functions an immediate consequence of the non-