

Some Results on Power Bounded Operators

M. RADJABALIPOUR

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It is well known that an operator T on a Hilbert space which is power bounded (i.e., $\|T^n\| < k$, for $n = 0, 1, 2, \dots$) need not even be similar to a contraction [1], [2]. Here we show that, given some supplementary hypotheses, T is necessarily similar to a unitary operator.

The theorems were found independently of the recent papers [4], [5]. The direct proofs will appear in my thesis. Here I give shorter proofs by relying on results from [4], [5].

Throughout the paper we will be considering bounded operators T on a Hilbert space H endowed with the inner product norm $\|\cdot\|$. The usual notations $R_z(T)$, $N(T)$, and $R(T)$ will be used for $(z - T)^{-1}$, the null space of T , and the range of T respectively.

Let M_i , $i = 1, 2, \dots$, be a finite or an infinite sequence of closed subspaces of H . We say H is the direct sum of M_i , denoted by $\sum_i \oplus M_i$, if $H = \bigvee_i M_i$ and to each element x of H there corresponds a unique sequence (x_i) such that $x = \sum_i x_i$, $x_i \in M_i$, $i = 1, 2, \dots$. (The M_i 's need not be orthogonal.)

Theorem 1. *If T is a power bounded operator on a Hilbert space H , and $H = \bigvee_{|z|=1} N(z - T)$, then T is similar to a unitary operator.*

Note. Following terminology in [4], an operator T is called C -full in case $H = \bigvee_{|z|=1} N(z - T)$.

Proof of Theorem 1. Define a new norm $|\cdot|$ on H by $|x| = \sup_{n \geq 0} \|T^n x\|$, $x \in H$. It is easy to see that the two norms $\|\cdot\|$ and $|\cdot|$ are equivalent and $(H, |\cdot|)$ is a reflexive Banach space. Moreover $|T| \leq 1$, which implies that T , as an operator on $(H, |\cdot|)$, is a C -full contraction.

It follows from [4, Th. 2] that T is a unitary operator, whence $|T^n| = 1$ for $n = 0, \pm 1, \pm 2, \pm 3, \dots$.

But now the equivalence of the two norms implies that $\|T^n\| < k$, where k is a positive constant and $n = 0, \pm 1, \pm 2, \dots$. Thus by the criterion of Sz.-Nagy [7], T is similar to a unitary operator. The proof of the theorem is complete.