

The Structure of Intertwining Isometries

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Introduction. The object of this paper is the examination of the spectral theory of a class of partial isometries that intertwine self-adjoint operators.

We succeed, among other things, in giving a complete and perfectly explicit description of the unitary invariants of all isometries W for which

$$WV = (V + D)W$$

where V is a bounded self-adjoint operator and D is a positive operator of one dimensional range provided that the least subspace which contains the range of D and which reduces V is invariant for W . We also obtain a similar result for the minimal unitary dilation of W in case W is not an isometry but a partial isometry, provided the initial space of W is contained in $\mathcal{H}_{ac}(V)$.

Our method consists in making an application of the so-called Determining Function theory developed in Pincus [1], [2], [3], [4], [5], but we restrict ourselves here to a discussion of the case where D has one dimensional range even though our methods lead to more general results.

Each intertwining isometry W is characterized up to unitary equivalence by a positive function $g(\lambda, \gamma)$ defined on the cylinder $\sigma(V) \times \{|z| = 1\}$.

This function, called the principal function, is in fact the complete unitary invariant of the triple $(W; V, V + D)$. When W is isometric this function takes on only the values zero and one.

We shall show that the spectral multiplicity of the unitary part of W , in this case, is easily obtained from a knowledge of the principal function. Furthermore, all partial isometries which satisfy the intertwining relation satisfy a basic relation involving the principal function.

If $g(\lambda, \gamma)$ is the principal function corresponding to the triplet $(W; V, V + D)$, then we prove that

$$(0.1) \quad \delta(\lambda) = \frac{1}{2\pi} \int_0^{2\pi} g(\lambda, e^{i\theta}) d\theta,$$

where $0 \leq \delta(\lambda) \leq 1$ and $\delta(\lambda)$ is the "phase shift" corresponding to the perturbation problem $V \rightarrow V + D$, *i.e.*, the scattering operator can be represented by a multiplication operator, $s(\lambda) = e^{-2\pi i \delta(\lambda)}$.

We shall show, further, that any set on the cylinder $\sigma(V) \times$ unit circle whose