

Extreme Points of Multifunctions

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1. Introduction and notation. Recently C. Castaing [C-1] showed that under suitable restrictions the measurability of a multifunction F implies the measurability of the multifunction F^0 whose value at t is the set of extreme points of $F(t)$. From this he showed that the integrals of these two multifunctions are equal. The purpose of this paper is to continue the investigation of the relationship between the extreme point multifunction F^0 and the original multifunction. In particular, we show that smoothness (some kind of continuity) of F does not have to be inherited by F^0 . However, in analogy to the well known result that, for a compact convex metrizable subset A of a locally convex space, the set A^0 of extreme points of A is a G_δ set, we show that the extreme point multifunction F^0 of an upper semicontinuous multifunction F has a G_δ graph. This in turn allows us to show, under different conditions than employed by Castaing, that the extreme point multifunction has a G_δ graph provided an appropriate set of arbitrarily small measure is deleted. This shows that the measurability of F implies the measurability of F^0 . The converse is also true as has been pointed out by Castaing [C-2]. The above results constitute the bulk of Section 2, where it is assumed that the domain is (at least) a locally compact Hausdorff space T equipped with a positive Radon measure μ .

In Section 3 we require only that T be a measurable space, but increase the requirements on the range of our multifunction F by requiring F to have as its values subsets of a Euclidean space. We show that, if the measurable space T can be equipped with a complete measure, then F^0 is measurable if and only if F is measurable. The technique used here is, up to a certain point, that introduced by Castaing [C-1].

Finally, in the last section we apply our results to give another proof of the bang-bang principle for control systems of the form $x' = A(t)x + \varphi(t, u)$. Our result here is essentially the same as that obtained by Olech [O-2] who uses a different and, in some ways, more general technique. We are able to say a little more about the measurability of a multifunction constructed in the course of the proof, but Olech's result follows from a more general technique.