

# The Initial-Boundary Value Problem for the Navier-Stokes Equations with Data in $L^p$

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**Introduction.** Let the  $L^{p,q}$  norm of a function  $f(t, x)$  of  $t$ , time, and a spatial variable  $x$  be given by

$$\|f\|_{p,q} = \left( \int_0^T \left( \int_{\text{space}} |f(t, x)|^p dx \right)^{q/p} dt \right)^{1/q}.$$

A condition of regularity for solutions of nonstationary Navier-Stokes equations [4] is to require that the solution  $u = (u^1, \dots, u^n)$  belongs to the class  $L^{p,q}$ , with  $n/P + 2/Q = 1$  and  $n < P < \infty$ . For the nonlinear initial-boundary value problem in a half-space, it will be shown that requiring the boundary data  $a$  to be in an  $L^{p,q}$  space, with  $(n-1)/p + 2/q = 1$  and  $n-1 < p < \infty$ , will yield a solution  $u \in L^{p,q}$ . The analytic tool which gives relations of the form (space dimension)/ $p + 2/q = 1$  is the Sobolev Embedding Theorem.

Specifically, the following problem is considered: in  $[0, T) \times \mathbf{R}_+^n$ , one seeks a vector valued function  $u(t, x, y) = (u^1, \dots, u^n)$  and a scalar valued pressure function  $P(t, x, y)$  which satisfy

$$\begin{aligned} (0.1) \quad & u_t - \Delta u = -(u \cdot \nabla)u - \nabla P \quad \text{in } (0, T) \times \mathbf{R}_+^n, \\ & \nabla \cdot u = 0 \quad \text{in } (0, T) \times \mathbf{R}_+^n, \\ & u(0, x, y) = g(x, y) \quad \text{in } \mathbf{R}_+^n, \quad \text{with } \nabla \cdot g = 0, \\ & u(t, x, 0) = a(t, x) \quad \text{in } (0, T) \times \mathbf{R}^{n-1}. \end{aligned}$$

Here  $\Delta$  is the Laplace operator  $\sum_{i=1}^{n-1} \partial^2/\partial x_i^2 + \partial^2/\partial y^2$  and the symbol  $\nabla$  is the gradient in the space variables  $(x, y)$  only;  $\nabla \cdot u$  is the divergence  $\sum_{i=1}^{n-1} \partial u^i/\partial x_i + \partial u^n/\partial y$ . The Euclidean real space of dimension  $k$  is denoted by  $\mathbf{R}^k$ ; let  $\mathbf{R}_+^k$  be the open half-space of points  $\{(x, y) : x \in \mathbf{R}^{k-1}, y > 0\}$ .

We suppose that the initial data  $g(x, y) \in L^r(\mathbf{R}_+^n)$ ,  $n < r < \infty$ , and that  $\nabla \cdot g = 0$  in the sense of distributions; no condition of integrability is assumed