

Wings in Analytic Sets

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The notion of a regularly stratified analytic set V was introduced by Whitney in [1].

A stratification of V is a set of submanifolds (M_i) of V , whose union is V and satisfy some mild topological restrictions. If M_1 and M_2 are two strata and $p \in M_2 \subset \bar{M}_1$, then we say that M_1 is a (resp. b) regular over M_2 if for any sequence $(q_i) \subset M_1$, $(q_i) \rightarrow p$, such that the tangent planes $(T(M_1, q_i))$ approach a limit plane τ we have $T(M_2, p) \subset \tau$ (resp. the "limiting direction" of the secants from q_i to p is contained in τ). We say that the stratification is a -regular (resp. b -regular) if the above condition holds, for all pairs of strata, at all such points.

In [2] Whitney showed that any analytic set admits such a stratification. In the course of his survey of the theory of stratified sets in [3] Wall gives a short description of Whitney's proof. He observes that the essential point is the proof of the following result. Let M_1 and M_2 be as above, then $S_a = \{x \in M_2 : \bar{M}_1 \text{ is not } a \text{ regular over } M_2 \text{ at } x\}$ is an analytic set of codimension ≥ 1 in M_2 , and similarly for S_b . If M_2 is a point, Wall observes that S_a and S_b are empty. This result follows immediately from the Bruhat-Cartan-Wallace curve section lemma [4]. In general however one must use a generalization of this, the "wing lemma", proved in [2]. Unfortunately this wing lemma is both weaker and more difficult to prove than the curve section lemma. Its weakness complicates the proof in [2] of the result mentioned above. The purpose of this note is to give a strengthened version of the wing lemma and then to present a simplified proof of the result on the dimension of S_a and S_b . This will be done in Sections 2 and 3 respectively. In Section 1 we briefly review some facts about stratifications as background for our results.

Section 1. Let V be an analytic set of dimension n . A partition of V is a set of submanifolds (M_i) of V which are pair-wise disjoint, locally finite, and have $V = \bigcup_i M_i$. For us submanifolds are not necessarily closed subsets of V . We say (M_i) is strict if for any i

1.1. \bar{M}_i and $\bar{M}_i - M_i$ are analytic subsets of V .