

Excluded Minors of Combinatorial Geometries

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In this paper we present a theorem concerning the exclusion of minors from single-point extensions of combinatorial geometries (simple matroids). After a review of the theory developed by H. H. Crapo in [1], the theorem is proved and then specialized to characterize those extensions of binary geometries that are binary. Our terminology follows that of [2].

Introduction. A pregeometry G on a set S is a closure relation $A \mapsto \bar{A}$ on S satisfying the properties:

- (i) For any $a, b \in S$ and for any $A \subseteq S$, if $a \in \overline{A \cup b}$ and $a \notin A$, then $b \in \overline{A \cup a}$;
- (ii) For any $A \subseteq S$ there exists a finite set $B \subseteq A$ such that $\bar{B} = \bar{A}$.

A combinatorial geometry (or geometry) on the set S is a pregeometry such that (iii) $\bar{\emptyset} = \emptyset$ and $\bar{a} = a$, any $a \in S$.

A closed set of a pregeometry is called a *flat*.

Many authors refer to pregeometries as matroids and to combinatorial geometries as simple matroids. If we wish to indicate that S is the underlying set of G we write $G(S)$.

It is known that the flats of a pregeometry, ordered by inclusion, form a geometric lattice with point set $\{\bar{a} : a \in S, \bar{a} \neq \bar{\emptyset}\}$. Conversely, if L is a geometric lattice with S as its point set, the closure relation $\bar{A} = \{a \in S : a \leq \sup A\}$, $A \subseteq S$, defines a combinatorial geometry on S . With any pregeometry we therefore canonically associate the combinatorial geometry determined by its lattice of flats. The pregeometry and its associated geometry have isomorphic lattices of closed sets. See [2, Chapter 2] for proofs.

With any pregeometry $G(S)$ we associate a rank function defined on the power set of S . The rank of $A \subseteq S$, $r(A)$, is one less than the cardinality of a maximal chain from $\bar{\emptyset}$ to \bar{A} in the lattice of flats of G . The rank function is semi-modular: $r(A \cup B) + r(A \cap B) \leq r(A) + r(B)$, any $A, B \in S$. A flat A is said to *cover* flat B if $B \subset A$ and $r(A) = r(B) + 1$. Suppose $r(S) = n$. If A is a flat with $r(A) = 2$ or $n - 1$ or $n - 2$, then A is called a *line*, *copoint* or *coline*, respectively.