

The Spectral Multiplicity of a Multiplication Operator

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Let (X, \mathcal{G}, μ) be a separable probability space, let ϕ be in $L^\infty(\mu)$, and let M_ϕ be the bounded normal operator on the Hilbert space $L^2(\mu)$ given by $M_\phi(f) = \phi f$. In this paper we give an explicit calculation of the spectral multiplicity function of M_ϕ in terms of ϕ and μ . It is surprising that this does not seem to have been done previously except in special cases which are part of the folklore of operator theory. These special cases suggest that the multiplicity of a point y is equal to the number of points in the pre-image $\phi^{-1}(y)$. This is literally true only in very special circumstances; an example in Section 5 shows that it is not true for a general continuous function on the unit interval with μ equal to Lebesgue measure. However, it is true in a rather general topological setting if one adopts the notion of essential pre-image introduced for a special case in [6]. To compute the multiplicity without any topological assumptions on X , we give a way of counting the number of points that "ought" to be in the essential pre-image without actually locating the points.

This paper is divided into seven sections. Section 1 states the direct integral representation of a normal operator and defines spectral measure and multiplicity function. This material is all well known and proofs can be found in [2]. Section 2 contains a theorem in integration theory on disintegration of measures. It is only after considerable hesitation that we include a proof of this theorem because it is similar to other theorems in the literature, see for instance [7]. But nowhere could we locate the precise theorem that we needed and furthermore its inclusion makes the paper essentially self-contained. Section 3 is a construction of the direct integral for a multiplication operator M_ϕ where ϕ is a Borel measurable function on a locally compact separable metric space. A construction along these lines was carried out by R. G. Douglas [3] for a continuous function on a compact metric space. A different construction which accomplishes the same thing was recently shown to us by James Howland. In Section 4 we define the essential pre-image and show how it can be used to compute the multiplicity of M_ϕ . The development here is based on a theorem of