

Interpolation Theorems for Compact Operators

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§1. Introduction. In [3] and [7] interpolation theorems of the following kind were considered:

Each linear operator which is a bounded linear operator from X_i to Y_i , $i = 1, 2$, and which is also a compact operator in one of the two cases has a unique extension to a compact operator from X to Y .

Such pairs (X, Y) are said to be compact intermediate for the interpolation segment $\sigma = [(X_1, Y_1), (X_2, Y_2)]$. All spaces in [3] were L^p spaces, while in [7], $X_1 = Y_1 = L^p$, $X_2 = Y_2 = L^q$ (or dually, $X_1 = Y_1 = L^1$, $X_2 = Y_2 = L^q$), and $X = Y$ were rearrangement invariant spaces. We give extensions and generalizations of theorems of this kind using Calderón's operator S_σ in Sections 3 and 4 and the compression (or dilation) operator σ_σ in Sections 5 and 6.

In Section 2 we define rearrangement invariant Banach function spaces, the fundamental function of such a space, and Calderón's operator S_σ and give their various properties. Several theorems from [11] are stated for completeness.

A sufficient condition is given in Section 3 for weak compact (and hence compact) interpolation in terms of a function F which depends in a simple manner on the fundamental functions of the six spaces involved. This condition is shown to be necessary if the intermediate spaces are extremal. In Section 4 we obtain necessary and sufficient conditions for the scale $\Lambda_\alpha(X)$, $0 < \alpha \leq 1$, to be weak compact intermediate for an interpolation segment.

In Section 5 we get a sufficient condition for compact interpolation involving the Boyd indices of the six spaces when the underlying measure space is infinite or when the slope of the interpolation segment is positive. This condition is necessary when $X = Y$ and $\sigma = [(L^p, L^p), (L^q, L^q)]$. Section 6 deals with the remaining case, *i.e.*, finite measure space and an interpolation segment with negative slope. This requires use of an "adjoint" operator and a duality argument.

The results of §3 and §4 deal with weak compact interpolation and do not give Krasnosel'skii's theorem, whereas Theorems 3 and 4 of §5 and §6, respectively, do. Also Theorems 1 and 2 do not follow from the results of §5 and §6 since the operator S_σ does not belong to the class of compact operators in