

The Phase Shift and Singular Measures

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1. Introduction. A theorem of Herglotz-Riesz in a form due to R. Nevanlinna [AG; p. 7] implies the existence of a one-to-one correspondence between the set of analytic functions in the upper half-plane with positive imaginary part and value one at infinity and the set of all positive, finite Borel measures on the real line. The correspondence is given by:

$$(1.1) \quad \phi(\ell) = 1 + \int_{-\infty}^{\infty} \frac{d\mu(t)}{t - \ell},$$

where ϕ is an analytic function and μ is the Borel measure. By choosing the principal determination of the logarithm, one also obtains the exponential representation of such functions:

$$(1.2) \quad \phi(\ell) = \exp \left[\int_{-\infty}^{\infty} \frac{\xi(t)}{t - \ell} dt \right],$$

where ξ is a Lebesgue summable function (the so called phase shift) for which $0 \leq \xi(x) \leq 1$ and

$$\int_{-\infty}^{\infty} \xi(x) dx = \int_{-\infty}^{\infty} d\mu(x).$$

In [CP], Carey and Pincus demonstrated how the function ξ arises as the complete unitary invariant of a one-dimensional perturbation problem consisting of a triplet of operators $\{V, D, W_{\pm}\}$ where V and D are self adjoint operators acting on a Hilbert space and W_{\pm} are wave operators. In particular, it was shown how the von Neumann spectral multiplicity function for W_{\pm} may be explicitly computed in terms of ξ . The ξ function essentially occurs as a perturbation determinant which has been studied by various authors interested in the problem of perturbation of spectra ([D1] & [PP]).

In the present paper an investigation of the local oscillatory behavior of ξ is carried out in an effort to determine when the measure μ has a nontrivial singular component. A necessary and sufficient condition is given in terms of