

Derivations, Plücker Relations, and the Numerical Range

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I. Introduction. Let V be an n -dimensional unitary space with $A \in \text{Hom}(V, V)$. If $1 \leq r \leq m \leq n$ and x_1, \dots, x_m is an orthonormal (o.n.) set in V , then define

$$(1) \quad f_{r,m}(x_1, \dots, x_m) = \sum_{\omega \in Q_{r,m}} \det [(Ax_{\omega(i)}, x_{\omega(j)})].$$

In (1) $Q_{r,m}$ is the sequence set consisting of all $\binom{m}{r}$ sequences ω of length r chosen from $1, \dots, m$ which satisfy $\omega(1) < \dots < \omega(r)$. As the x_1, \dots, x_m run over all o.n. sets in V , $f_{r,m}(x_1, \dots, x_m)$ describes a set of complex numbers in the plane which we shall denote by $W_{r,m}^n(A)$.

The classical Toeplitz–Hausdorff result asserts that the numerical range $W_{1,1}^n(A)$ is a convex set, *i.e.*, the set of numbers (Ax, x) , $\|x\| = 1$, is convex in the plane. (A simple inductive proof of this result is found in [4].) In [5] P. R. Halmos conjectured that $W_{1,m}^n(A)$ is always convex. R. C. Thompson [11] proved this when A is normal, and simultaneously [1] C. A. Berger obtained the result in general. Just recently J. de Pillis [2] proved that if $x_1, \dots, x_m, y_1, \dots, y_m$ is an o.n. set, *i.e.*, $\langle x_1, \dots, x_m \rangle$ and $\langle y_1, \dots, y_m \rangle$ are orthogonal subspaces of V , then the line segment joining $f_{r,m}(x_1, \dots, x_m)$ and $f_{r,m}(y_1, \dots, y_m)$ is entirely contained in $W_{r,m}^n(A)$. de Pillis also conjectures that $W_{r,m}^n(A)$ is in fact always convex. However, the Thompson paper contains an example [11; p. 225] which shows that for $n = 4, r = m = 2$, $W_{2,2}^4(A)$ is definitely not convex.

The purpose of the present paper is to prove the following result.

Theorem 1. *If $1 \leq r \leq n - 1$ and $A \in \text{Hom}(V, V)$, then $W_{r,n-1}^n(A)$ is convex. If $2 \leq r < n - 1$, then there exists a normal $A \in \text{Hom}(V, V)$ such that $W_{r,r}^n(A)$ is not convex.*

The second assertion in the above result settles the question (in the negative) raised by de Pillis [2; p. 780].