

# Some Results on Non-quasitriangular Operators

CONSTANTIN APOSTOL, CIPRIAN FOIAȘ &  
LÁSZLÓ ZSIDÓ

*Communicated by the Editors*

Let  $\mathfrak{H}$  be a separable-definite dimensional, complex Hilbert space and let  $\mathfrak{L}(\mathfrak{H})$  denote the algebra of all bounded, linear operators acting in  $\mathfrak{H}$ . Let  $\mathfrak{S}(\mathfrak{H})$  denote the directed set of all finite-dimensional subspaces (= closed linear manifolds) of  $\mathfrak{H}$ . If  $\mathfrak{A}$  is a subspace in  $\mathfrak{H}$ , then  $\mathfrak{A}^\perp$  is the orthocomplement of  $\mathfrak{A}$  and  $P_{\mathfrak{A}}$  is the orthogonal projection of  $\mathfrak{H}$  onto  $\mathfrak{A}$ . For any  $T \in \mathfrak{L}(\mathfrak{H})$  set

$$q(T) = \lim_{\mathfrak{A} \in \mathfrak{S}(\mathfrak{H})} \|P_{\mathfrak{A}^\perp} T P_{\mathfrak{A}}\| = \sup_{\mathfrak{A} \in \mathfrak{S}(\mathfrak{H})} \left( \inf_{\substack{\mathfrak{A} \subset \mathfrak{B} \\ \mathfrak{B} \in \mathfrak{S}(\mathfrak{H})}} \|P_{\mathfrak{B}^\perp} T P_{\mathfrak{B}}\| \right),$$

$$Q(T) = \overline{\lim}_{\mathfrak{A} \in \mathfrak{S}(\mathfrak{H})} \|P_{\mathfrak{A}^\perp} T P_{\mathfrak{A}}\| = \inf_{\mathfrak{A} \in \mathfrak{S}(\mathfrak{H})} \left( \sup_{\substack{\mathfrak{A} \subset \mathfrak{B} \\ \mathfrak{B} \in \mathfrak{S}(\mathfrak{H})}} \|P_{\mathfrak{B}^\perp} T P_{\mathfrak{B}}\| \right).$$

In [6], Halmos initiated the study of quasitriangular operators, namely of those  $T \in \mathfrak{L}(\mathfrak{H})$  for which  $q(T) = 0$ . The case of operators  $T$  for which  $Q(T) = 0$  was clarified by Douglas and Percy, [3]; they showed that such a  $T$  is thin, *i.e.*,  $T = \lambda + K$ , with  $\lambda$  scalar and  $K$  compact.

In this paper we try to illustrate the point of view that the degree of simplicity of  $T$  is directly proportional with  $q(T)/Q(T)$ , giving an insight into the structure of those  $T$  for which  $q(T)/Q(T) > \frac{1}{2}$ . We show, in particular, that the adjoint of an operator satisfying this last condition has non-void point spectrum (see [7], Problem X).

In the first section we characterize  $Q(T)$ , showing that it is the distance of  $T$  to the thin operators. In the second section we give some evaluation for  $q(T)$ , which allows us to obtain in the third section a structure theorem for operators  $T$  with  $q(T)/Q(T) > \frac{1}{2}$ . A consequence of this result is the following generalization of the theorem of Douglas and Percy ([3], Theorem 2):  $\lim_{\mathfrak{A} \in \mathfrak{S}(\mathfrak{H})} \|P_{\mathfrak{A}^\perp} T P_{\mathfrak{A}}\|$  exists (*i.e.*,  $q(T) = Q(T)$ ) if and only if  $T = \lambda V + S$ , with  $\lambda$  scalar,  $V$  a non-