

# Non-Linear Extremal Problems in $H^\infty$

STEPHEN FISHER

*Communicated by G. SPRINGER*

Let  $R$  be a finite open Riemann surface with boundary  $\Gamma$  and let  $H^\infty(R)$  be the algebra of bounded holomorphic functions on  $R$ . In this paper we consider the following extremal problem: fix distinct points  $a$  and  $b$  in  $R$  and let

$$(*) \quad M(b; a) = \sup \{|h(b)| : h \in H^\infty(R), |h| \leq 1, h(a) = 0,$$

and  $h$  has no other zeros on  $R\}$ .

It is clear that a solution to  $(*)$  exists; we investigate the properties of such solutions. A similar problem, where the restraint on the zeros of the competing functions on  $R - \{a\}$  is removed, has been investigated by L. Ahlfors [A1], H. Royden [R], and others. In that case, the class of competing functions is convex, the solution is unique, is analytic across  $\Gamma$ , and has modulus one on  $\Gamma$ . In our problem, the class of competing functions is not convex, a fact which adds difficulties to determining the solutions. However, we prove that every solution to  $(*)$  is analytic over  $\Gamma$  except possibly at a finite number of points, the number of which depends directly on  $R$ , and each solution has modulus one at those points of  $\Gamma$  at which it is analytic. We also show that there can be more than one solution to  $(*)$ .

Section 1 contains information on positive harmonic functions on  $R$  which is needed in Section 2 where we investigate the nature of solutions to  $(*)$ . In Section 3, we briefly discuss a more general setting for the non-linear extremal problem  $(*)$  and a related extremal problem.

**Section 1. Positive harmonic functions on  $R$ .** Let  $m$  be harmonic measure on  $\Gamma$  for  $b$ ; if  $z$  is any point of  $R$ , then the Radon–Nikodym derivative of  $m_z$ , the harmonic measure for  $z$ , with respect to  $m$  is a smooth function  $P_z$  on  $\Gamma$ . If  $\mu$  is a finite regular Borel measure on  $\Gamma$  we define the harmonic extension of  $\mu$  to  $R$  by the rule

$$(1.1) \quad \tilde{\mu}(z) = \int_{\Gamma} P_z d\mu.$$