

# *Two Sublattices of Weighted Shift Invariant Subspaces*

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## **Introduction.**

1. A weighted shift is an operator in Hilbert space sending

$$(1.1) \quad e_n \text{ to } a_{n+1}e_{n+1},$$

where  $\{e_n\}_0^\infty$  is an orthonormal basis and  $\{a_n\}_0^\infty$  is a never-zero sequence.

The unweighted shift  $S$  (all  $a$ 's = 1) has been much studied since Beurling proved in [1] that

- 1)  $S$  restricted to an invariant subspace is unitarily equivalent to  $S$  (acting on the whole space) and
- 2) every invariant subspace of  $S$  is the range of an isometry commuting with  $S$ .

There has been some interest in whether some generalization of these properties still hold for weighted shifts. Halmos has asked in [4], Problem 2, whether every part of a weighted shift is similar to a weighted shift (a part being a restriction to an invariant subspace). In Section I it is shown that this possible generalization of 1) is false, and also that there is a weighted shift with an invariant subspace not the closure of the range of any operator commuting with the shift, thus disposing of a weak possible generalization of 2).

The main tool in finding these counterexamples is a complete description of the lattice of finite co-dimensional shift invariant subspaces. Information is extracted from [3], Section 9, which describes all such subspaces for two-sided shifts in Banach space, but does not make the lattice structure explicit. The absence of a proof will signify that the proof is left as an easy exercise.

In Section II it is shown that an operator commuting with an invertible bilateral weighted shift is the strong limit of polynomials in the shift and its inverse, that the commutant is naturally isomorphic to a ring of bounded measurable functions on the spectrum of the shift, and that the invariant subspaces with invariant complements form a lattice isomorphic with a lattice of measurable subsets of a circle invariant under rotation.