

# *A Characterization of $T_3$ -Spaces*

DARRELL W. HAJEK

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A well known result in general topology is the characterization of  $T_{3\frac{1}{2}}$  (i.e., completely regular Hausdorff) spaces as precisely those which can be embedded in compact Hausdorff spaces. In this paper, we establish a similar characterization for  $T_3$  (i.e., regular Hausdorff) spaces. We then utilize this characterization to prove that if a continuous function has a  $T_3$  range and a Wallman extension, then it must be a *WO* function.

By an extension of a topological space  $X$  we will mean a pair  $\langle Y, \varphi \rangle$  where  $Y$  is a topological space and  $\varphi$  is a dense embedding of  $X$  in  $Y$ . Clearly the above characterization of  $T_{3\frac{1}{2}}$  spaces can be rephrased: A space is  $T_{3\frac{1}{2}}$  if and only if it has a compact Hausdorff extension. We will say that an extension  $\langle Y, \varphi \rangle$  of a space  $X$  is relatively Hausdorff provided that for any point  $x \in X$  and any point  $y \in Y$  distinct from  $\varphi(x)$ , there exist disjoint open subsets  $U$  and  $V$  of  $Y$  containing  $\varphi(x)$  and  $y$  respectively.

**Proposition 1.** *If a space  $X$  has a compact relatively Hausdorff extension, then  $X$  is  $T_3$ .*

*Proof.* It is obvious that any space which has a relatively Hausdorff extension must be a Hausdorff space. Suppose now that  $\langle Y, \varphi \rangle$  is a compact relatively Hausdorff extension of  $X$ , that  $A$  is a closed subset of  $X$ , and that  $x \in X \sim A$ . Since  $\varphi$  is an embedding,  $\text{cl}_Y(\varphi[A])$  does not contain  $\varphi(x)$ . Therefore, since  $\langle Y, \varphi \rangle$  is relatively Hausdorff, given any  $y \in \text{cl}_Y(\varphi[A])$  there exist disjoint open subsets  $U_y, V_y$  of  $Y$  such that  $y \in U_y$  and  $\varphi(x) \in V_y$ . Because  $\text{cl}_Y(\varphi[A])$  is a closed subset of a compact space it is compact.  $\{U_y : y \in \text{cl}_Y(\varphi[A])\}$  is an open cover of  $\text{cl}_Y(\varphi[A])$ , and so contains a finite subcover  $\{U_{y_i} : i = 1, \dots, n\}$ . Then  $\bigcup_{i=1}^n U_{y_i}$  and  $\bigcap_{i=1}^n V_{y_i}$  are disjoint open subsets of  $Y$  containing  $\varphi[A]$  and  $\varphi(x)$  respectively. Therefore  $\varphi^{-1}[\bigcup_{i=1}^n U_{y_i}]$  and  $\varphi^{-1}[\bigcap_{i=1}^n V_{y_i}]$  are disjoint open subsets of  $X$  containing  $A$  and  $x$  respectively. Hence  $X$  must be  $T_3$ .

Recall (from [1], [3], or [4]) that for any  $T_1$  space  $X$ , if  $W(X)$  denotes the collection of all ultrafilters in the lattice of all closed subsets of  $X$  with topology generated by  $\{C(A) = \{u \in W(X) : A \in u\} : A \text{ a closed subset of } X\}$  as a base