

Poincaré Metrics on Real Projective Space

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In this note we consider Riemannian metrics on 3-dimensional real projective space, P_3 , which are induced by Riemannian metrics on the 2-dimensional sphere, S_2 , (cf. below for details) and seek upper bounds for the quotient L^3/V , where (henceforth) L is the length of the shortest closed geodesic in the non-trivial free homotopy class of continuous, sectionally smooth $\xi: [0, 1] \rightarrow P_3$ and V is the volume of P_3 relative to the metric in question. It is conjectured (cf. [1–6] for background and related results) that for *all* Riemannian metrics on P_3 , $L^3/V \leq \pi$ the value achieved when the Riemannian metric on P_3 has constant sectional curvature, and that equality above characterizes the Riemannian metric of constant sectional curvature on P_3 . We are content to limit the class of Riemannian metrics to much less than the class of all Riemannian metrics as there are sufficiently many difficulties yet to be resolved in this restricted case.

Our class of Riemannian metrics is constructed as follows: the unit tangent bundles of different Riemannian metrics on the 2-sphere, S_2 , are diffeomorphic, and the unit tangent bundle of the Riemannian metric of constant curvature 1 on S_2 can be explicitly written as $SO(3)$, the special orthogonal group acting on 3-dimensional Euclidean space. But $SO(3)$ is diffeomorphic to P_3 ([16, p. 115]). For any Riemannian metric g on S_2 one constructs in the standard manner the three global, pointwise linearly independent, differential 1-forms, $\omega_1, \omega_2, \omega_{12}$ on P_3 now viewed as the oriented frame bundle of S_2 , where ω_1, ω_2 are the canonical forms of the bundle, and ω_{12} is the Levi-Civita connection form on P_3 of the Riemannian metric g . The forms $\omega_1, \omega_2, \omega_{12}$ satisfy the Cartan structure equations

$$d\omega_1 = -\omega_2 \wedge \omega_{12},$$

$$d\omega_2 = \omega_1 \wedge \omega_{12},$$

$$d\omega_{12} = -K\omega_1 \wedge \omega_2,$$

where K is the Gaussian curvature of g on S_2 (lifted to P_3 via the standard projection of the unit tangent bundle). The induced Riemannian metric on P_3