

# *Lipschitz Spaces and Exponentially Integrable Functions*

C. J. NEUGEBAUER

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1. We let  $\mathbf{R}^n$  be the Euclidean  $n$ -space, and we consider the Lipschitz spaces  $\Lambda(\sigma; p, s; \mathbf{R}^n)$  as defined in [2, 5]. Our main result will imply an imbedding of the Lipschitz spaces  $\Lambda(n/p; p, s; \mathbf{R}^n)$  into the class  $\text{exp}(L^s)$  consisting of all functions  $f: \mathbf{R}^n \rightarrow \mathbf{R}$  with compact support satisfying  $\int \exp(\alpha |f(x)|^s) dx < \infty$  for some  $\alpha > 0$ ,  $1/s + 1/s' = 1$ , where the integration is extended over the support of  $f$ .

In view of the fundamental inclusion relationships between the space  $\Lambda(\sigma; p, s; \mathbf{R}^n)$  and the space of Bessel potentials  $L_p^\sigma(\mathbf{R}^n)$  (see [2, 5]), this result extends, for  $p \geq 2$ , the inequality obtained by Strichartz [4].

The proof is accomplished by estimating  $\|f\|_q$  in terms of the  $L^p$ -modulus of continuity of  $f$ , and the proof of this estimate is an  $n$ -dimensional adaptation and refinement of an argument of Storozhenko [3] whose main result, for  $n = 1$ , is the exponential integrability of  $f$  if its  $L^p$ -modulus of continuity  $\omega_p(f; h) = O(h^{1/p})$ .

2. For  $f \in L^p(\mathbf{R}^n)$  we define

$$\omega_p(f; h) = \sup_{|t| \leq h} \left\{ \int_{\mathbf{R}^n} |f(x+t) - f(x)|^p dx \right\}^{1/p}.$$

It is well-known that  $\omega_p(f; h)$  is non-decreasing,  $\omega_p(f; h) = o(1)$  as  $h \rightarrow 0$ , and, if  $c$  is a positive integer,  $\omega_p(f; ch) \leq c\omega_p(f; h)$ .

Let  $Q(x; h)$  be the cube  $[x_1, x_1 + h] \times \cdots \times [x_n, x_n + h]$ , where  $x = (x_1, \dots, x_n)$ . If  $F$  is the integral of  $f$ , i.e.,  $F(x) = \int_0^{x_1} \cdots \int_0^{x_n} f$ , we define, in analogy with  $n = 1$ , the first and second differences of  $F$  as  $\Delta F(x; h) = \int_{Q(x, h)} f$ ,  $\Delta^2 F(x; h) = \Delta F(x; 2h) - 2^n \Delta F(x; h)$ .

**Lemma.** *Let  $f \in L^p(\mathbf{R}^n)$  for some  $p \geq 1$ . Then for  $q \geq p$ ,  $\|\Delta^2 F(\cdot; h)\|_q \leq ch^{n(1+1/q-1/p)} \omega_p(f; h)$ .*

*Proof.* We consider the case  $1 \leq p < \infty$ . The  $p = \infty$  situation is similar and easier. By an obvious change of variables we can write