

Automorphisms and Equivalence in von Neumann Algebras II

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1. Introduction. A problem which is often quite difficult to decide is whether the crossed product between a von Neumann algebra and a group of *-automorphisms is semi-finite or not. In the present paper we shall study an aspect of this problem by giving characterizations for the canonical image of a projection in the von Neumann algebra \mathfrak{R} to be finite in the crossed product \mathfrak{B} of \mathfrak{R} and the group G . Our approach is closely related to that in [8] in that we shall use the equivalence relation \sim_g on the projections in \mathfrak{R} defined in [8] by the automorphisms in G . The main result (Theorem 4.1) states that a projection in \mathfrak{R} is finite relative to the partial ordering defined by \sim_g if and only if its canonical image in \mathfrak{B} is a finite projection in \mathfrak{B} . As a consequence we obtain a characterization of such projections in terms of estimates (Theorem 4.2). When \mathfrak{R} is abelian these theorems can be viewed as results in ergodic theory. Note that if \mathfrak{R} is countably decomposable, then the equivalence relation \sim_g coincides with the one introduced by Hopf in [2], see [8]. In the final section we comment on the applications to ergodic theory and obtain a sharper version (Theorem 5.3) of Theorem 4.2, which appears to be new, and thus gives a new criterion for the existence of G -invariant σ -finite measures. The result is related to one of Markov [3, 4.8.7] on the existence of finite invariant measures (see also [7]).

2. The crossed product. Let \mathfrak{R} be a von Neumann algebra of operators on a Hilbert space \mathfrak{H} , and let G be a (discrete) group of *-automorphisms of \mathfrak{R} . The crossed product of \mathfrak{R} with G is the von Neumann algebra generated by \mathfrak{R} and G in a certain covariant representation. For convenience and in order to fix some notation we indicate here the construction of $\mathfrak{R} * G$ from [9]. This construction differs slightly from the one usually employed when G acts as a unitary transformation group on $B(\mathfrak{H})$ (see [1, Chap. I, §9] and [8, §3]), but gives an isomorphic result.