

# Products of Spaces with $m$ -Dense Subsets

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**Introduction.** Let  $X$  be an arbitrary topological space. In [3] Professor A. Wilansky introduced the notion of an  $m$  dense subset of  $X$  where  $m$  is a fixed infinite cardinal. We recall the definition here. (If  $A$  is a set, then  $c(A)$  will denote the cardinal of  $A$ .) A pair  $(D, I)$  is an  $m$ -density pair for  $X$  if the following are satisfied: (a)  $D \subset X$  and  $c(D) \leq m$ , (b)  $I$  is a totally ordered set with  $c(I) \leq m$ , (c) each point in  $X$  is the limit of a net in  $D$  which is directed by  $I$ . If  $X$  has an  $m$ -density pair, then  $X$  is said to have an  $m$ -dense subset. In [3] Professor Wilansky conjectured that if  $\{X_a : a \in A\}$  is a family of topological spaces each of which has an  $m$ -dense subset and if  $X = \prod \{X_a : a \in A\}$ , then  $X$  has an  $m$ -dense subset if  $c(A) < 2^m$  and has, in general, no  $m$ -dense subset if  $c(A) = 2^m$ . Assuming the generalized continuum hypothesis (G.C.H.), we show (Theorem 2.6) that if  $m$  is a regular cardinal and if each of the factors satisfies a condition (H), then the product space has an  $m$ -dense subset if  $c(A) < 2^m$ . (Condition (H) is satisfied by Hausdorff spaces and topological groups.) In Example 2.8 we show that if  $c(A) = 2^m$ , then the product space need not have an  $m$ -dense subset. Finally, we make some remarks concerning the conjecture when  $m$  is a singular cardinal.

As will be seen, our methods involve little topology. However, they draw heavily from the theory of ordinal and cardinal numbers. For this reason, we will begin with a summary of various results about ordinal numbers which we require. The reader should consult the elegant monograph of Takeuti and Zaring [2] for further details.

**§1. Ordinal and cardinal numbers.** If  $I$  and  $J$  are partially ordered sets, a *similarity transformation* of  $I$  into  $J$  is a one-one, order preserving function from  $I$  into  $J$ . If there is a similarity transformation of  $I$  onto  $J$ , then  $I$  and  $J$  are said to be *similar*. A subset  $K$  of  $I$  is said to be *cofinal* in  $I$  if for each  $i \in I$ , there is a  $k \in K$  with  $i \leq k$ . We write  $\text{cof}(I, J)$  if  $J$  is similar to a cofinal subset of  $I$ .

**Proposition 1.1.** *Let  $W$  be a well ordered set. Then there is a unique ordinal number  $\alpha$  such that  $W$  is similar to  $\alpha$ .*