

# *A Characterization of the Positive Cone of $\mathbf{B}(\mathfrak{h})$*

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$\mathfrak{h}$  is a complex Hilbert space with inner product  $(\cdot, \cdot)$ .  $\mathbf{B}(\mathfrak{h})$  is the algebra of all bounded linear operators on  $\mathfrak{h}$ .  $\mathbf{G}(\mathfrak{h})$  is the group of invertible elements of  $\mathbf{B}(\mathfrak{h})$ . The subset of accretive operators  $\mathbf{A}(\mathfrak{h})$  is defined by

$$(1) \quad \mathbf{A}(\mathfrak{h}) = \{T \in \mathbf{B}(\mathfrak{h}) \mid \operatorname{Re} T \equiv \tfrac{1}{2}(T + T^*) \geq 0\},$$

while

$$(2) \quad \mathbf{P}(\mathfrak{h}) = \{T \in \mathbf{A}(\mathfrak{h}) \mid T = T^*\}$$

is the cone of non-negative self-adjoint operators. The principal result of this paper is the following characterization of  $\mathbf{P}(\mathfrak{h})$ :

**Theorem 1.**  $T \in \mathbf{P}(\mathfrak{h})$  if and only if  $T^n \in \mathbf{A}(\mathfrak{h})$ ,  $n = 1, 2, \dots$ .

This theorem seems to have first been proved by C. R. Johnson [9] in the case in which  $\mathfrak{h}$  is finite dimensional. His proof is decidedly matricial in character, whereas our proof in the general case makes fundamental use of a mapping theorem for numerical ranges due to T. Kato [10]. In Section 5, we observe that the Kato theorem is in reality a mapping theorem for spectral sets.

An immediate consequence of Theorem 1 is that the only semi-groups of accretive operators are semi-groups of non-negative operators which of necessity are commutative. In Sections 2 and 3 we present the main results along with some generalizations and some discussions of open problems. Section 4 contains an application of the methods of this paper to multiplicative commutators on  $\mathbf{G}(\mathfrak{h})$  for which we give an improvement of a theorem of Putnam [11].

**§1. Notation and preliminaries.** For  $r > 0$ ,  $\Delta_r = \{\lambda \in \mathbf{C} \mid |\lambda| \leq r\}$  is the closed disk of radius  $r$  centered at 0.  $\Pi = \{\lambda \in \mathbf{C} \mid \operatorname{Re} \lambda \geq 0\}$  is the closed right half plane. For  $0 < \alpha < 2$ ,  $S_\alpha = \{\lambda \in \mathbf{C} \mid |\arg \lambda| \leq (\pi/2)\alpha\}$  is the closed wedge symmetric in the non-negative real axis centered at 0 and of opening  $\pi\alpha$ . If  $\Sigma \subset \mathbf{C}$ ,  $\Sigma^\circ$  is the interior of  $\Sigma$ ;  $\operatorname{Re} \Sigma = \{t \in \mathbf{R} \mid t + is \in \Sigma, s \in \mathbf{R}\}$ ;  $\operatorname{co} \Sigma$  is the closed convex hull of  $\Sigma$ ;  $\partial \Sigma$  is the boundary of  $\Sigma$ .