

# *The Cartan–Janet Theorem for Conformal Embeddings*

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**Introduction.** The Cartan–Janet Theorem [1], [6] asserts that any real analytic  $n$ -dimensional Riemannian manifold can be locally mapped by a real analytic isometric embedding into a Euclidean space of dimension  $\frac{1}{2}n(n + 1)$ . It is known (compare [4, Appendix 1]) that this result is dimension-wise the best possible; *i.e.*, there exist real analytic  $n$ -dimensional Riemannian manifolds which do not possess smooth local isometric embeddings in any Euclidean space of dimension strictly less than  $\frac{1}{2}n(n + 1)$ . If we seek a conformal embedding, however, we expect to be able to take a Euclidean space of one lower dimension. We will give two proofs that this is so, the first patterned upon Janet’s original paper, the second upon Cartan’s. It is hoped that the juxtaposition of the two proofs will help to clarify the relationship between the approaches of Cartan and Janet to the isometric embedding problem.

**1. Conformal embeddings: Janet’s approach.** We are interested in local results and take for the  $n$ -dimensional Riemannian manifold an open set  $U$  in  $\mathbf{R}^n$  together with some metric. Let  $H$  be a hypersurface in  $U$ ,  $p$  a given point of  $H$ , and  $f$  a map from  $H$  or  $U$  into  $E^N$  where  $E^N$  is Euclidean space of dimension  $N$ , *i.e.*,  $\mathbf{R}^N$  together with the usual metric and the usual coordinates. For  $f : H \rightarrow U$  an immersion we define  $\mathcal{O}_p(f(H))$  to be the osculating space of  $H$  at the point  $p$ , with  $H$  thought of as a submanifold of  $E^N$ . For any choice of local coordinates,  $\mathcal{O}_p(f(H))$  is the linear space in  $E^N$  spanned by  $\{f_{x_i}(p), f_{x_j x_k}(p)\}$ . We say  $f$  is nondegenerate if the osculating space is of maximal rank at  $p$  and hence in a neighborhood of  $p$ . We use  $\vdash$  both for the connection in  $U$  and in  $E^N$ , context makes clear which is meant. We shall always denote arbitrary elements in  $T_p(H)$  by  $X$  or  $Y$  and use  $Z$  for a unit vector in  $T_p(U)$  which is orthogonal to  $T_p(H)$ . Denote the second fundamental form of  $H$  in  $U$  by  $L$  and of  $f(H)$  in  $E^N$  by  $\bar{L}$ . So  $L(X, Y) = \langle Z, X \vdash \bar{Y} \rangle$ , for any extension of  $Y$ . Similarly for  $\bar{L}$ . Wherever necessary we use  $\langle \cdot, \cdot \rangle_p$  and  $L_p$  to indicate the dependence on a specific metric. The norm of a vector in  $E^N$  is written  $\|v\|$ .