

A Note on Non-Quasitriangular Operators II

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1. Introduction. Let \mathfrak{H} be a fixed, separable, infinite dimensional, complex Hilbert space, and let $\mathfrak{L}(\mathfrak{H})$ denote the algebra of all bounded linear operators on \mathfrak{H} . Let \mathcal{O} denote the directed set of all finite rank projections in $\mathfrak{L}(\mathfrak{H})$ under the usual ordering, and for each T in $\mathfrak{L}(\mathfrak{H})$ define

$$q(T) = \liminf_{P \in \mathcal{O}} \|(1 - P)TP\| \quad \text{and} \quad Q(T) = \limsup_{P \in \mathcal{O}} \|(1 - P)TP\|.$$

In [8], Halmos initiated the study of quasitriangular operators and proved that an operator T is quasitriangular if and only if $q(T) = 0$. In [3], Douglas and Pearcy employed the η -function of Brown and Pearcy (see [11], [3]) to prove that T is a *thin* operator (*i.e.*, an operator that is the sum of a scalar and a compact operator) if and only if $Q(T) = 0$. The functions q and Q were studied, respectively, by Apostol in [1] and by Foiaş and Zsidó in [6]. We appreciatively acknowledge access to preliminary versions of [1] and [6].

Let (QT) denote the set of all quasitriangular operators in $\mathfrak{L}(\mathfrak{H})$. One purpose of this note is to prove (in Section 2) the conjecture of Pearcy which states that if T is in $\mathfrak{L}(\mathfrak{H})$, then $q(T)$ is equal to the distance from T to the set (QT) , *i.e.*, $q(T) = \inf_{S \in (QT)} \|T - S\|$. In Section 3 we prove that if T is in $\mathfrak{L}(\mathfrak{H})$ and S is the compression of T to a subspace of finite codimension in \mathfrak{H} , then $q(T) = q(S)$ and $Q(T) = Q(S)$. A consequence is that if U is a nonunitary isometry in $\mathfrak{L}(\mathfrak{H})$, then the compression of U to a subspace of finite codimension in \mathfrak{H} is unitarily equivalent to a compact perturbation of U . Let (J) denote the ideal in $\mathfrak{L}(\mathfrak{H})$ of all compact operators, and let π denote the natural mapping of $\mathfrak{L}(\mathfrak{H})$ onto the quotient algebra $\mathfrak{L}(\mathfrak{H})/(J)$ (which is commonly known as the Calkin algebra). We prove that similarity in the Calkin algebra preserves quasitriangularity (a result which generalizes [4, Theorem 9]), and that unitary equivalence in the Calkin algebra preserves values of q . In Section 4 we prove that if T is an invertible operator such that $q(T) < 1/\|T^{-1}\|$, then $q(T^{-1}) \leq \|T^{-1}\|^2 \cdot q(T)/(1 - q(T)\|T^{-1}\|)$.

The following two lemmas will be used repeatedly in the sequel.

Lemma A. (*Apostol* [1]). *If A and B are in $\mathfrak{L}(\mathfrak{H})$, then $|q(A) - q(B)| \leq \|A - B\|$.*