

A Maximum Principle and Gradient Bounds for Linear Elliptic Equations

M. H. PROTTER & H. F. WEINBERGER

Communicated by DAVID GILBARG

1. Introduction. It is well known that if u is a harmonic function, then $|\text{grad } u|^2$ is subharmonic and therefore satisfies the maximum principle. This reasoning can be extended to solutions of second order elliptic equations with constant coefficients, but not to equations with variable coefficients.

The purpose of this paper is to obtain a simple extension to equations with variable coefficients of the above property of Laplace's equation.

Let u be a solution of the uniformly elliptic equation

$$(1.1) \quad Lu - cu \equiv a_{ij}u_{,ij} + b_i u_{,i} - cu = 0, \quad c \geq 0$$

in which $a_{ij}(x)$, $b_i(x)$, $c(x)$ are C^1 functions of $x = (x_1, x_2, \dots, x_N)$ in a domain D in \mathbf{R}^N . Here and throughout we use the convention that a repeated subscript indicates summation from 1 to N with respect to that index. Also, we write $u_{,i}$ for $\partial u / \partial x_i$, $u_{,ij}$ for $\partial^2 u / \partial x_i \partial x_j$, and so forth.

In Section 2, we show that there is a constant γ_0 which depends only on the coefficients in (1.1) and their derivatives such that for $\gamma \geq \gamma_0$ the quantity

$$|\text{grad } u|^2 + \gamma u^2$$

satisfies the maximum principle in D .

In Section 3, we show that if there is a nonnegative function ψ which satisfies the inequality $L\psi - c\psi \leq -1$ in D , then a similar and somewhat refined gradient estimate can be obtained for solutions of the inhomogeneous equation $Lu - cu = f$. This refined estimate is especially useful near boundary points where ψ vanishes. We also show how to obtain a modified result when c may be negative.

If the values of u on the boundary ∂D of D are prescribed C^2 functions and if ∂D is sufficiently smooth, one can bound the boundary values of $|\text{grad } u|$ by means of barrier functions. See, *e.g.*, [3, 8]. Our maximum principle can then be used to bound $|\text{grad } u|$ in D .