

Limit Theorems for Model Boltzmann Equations with Several Conserved Quantities

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1. Introduction. In this paper we consider a system of evolution equations of the form

$$(1.1) \quad p_t + vp_x = \frac{1}{\epsilon} Qp \quad (t > 0, x \in \mathbf{R}^1, v \in \{v_1, \dots, v_n\}, \epsilon > 0),$$

$$\lim_{\epsilon \downarrow 0} p(t, x, v) = f(x, v).$$

Here $p = p_\epsilon(t, x, v)$ and $Qp(t, x, v_i) = \sum_{j=1}^N q_{ij}p(t, x, v_j)$; $Q = (q_{ij})$ is a symmetric, negative semi-definite $N \times N$ matrix which has a d -dimensional null space \mathfrak{X} ($1 \leq d < N$); the numbers $\{v_i\}$ are real and distinct and $q_{ii} \neq 0$. The following pair of singular perturbation theorems is proved in Section 3.

Theorem 1. For each $(t, x, v) \in (0, \infty) \times \mathbf{R}^1 \times \{v_1, \dots, v_N\}$,

$$(1.2) \quad \lim_{\epsilon \downarrow 0} p_\epsilon(t, x, v) = \sum_{j=1}^d \langle f, e_0^{(j)} \rangle (x + \alpha_1^{(j)} t) e_0^{(j)},$$

where $\{e_0^{(j)}\}_{1 \leq j \leq d}$ is a special orthonormal basis for \mathfrak{X} and $\{\alpha_1^{(j)}\}_{1 \leq j \leq d}$ are real numbers given by the formulae $\alpha_1^{(j)} = -\langle e_0^{(j)}, ve_0^{(j)} \rangle$, $1 \leq j \leq d$. The notation $\langle \cdot, \cdot \rangle$ denotes the standard inner product in \mathbf{R}^{N-1} .

Theorem 2. For each $\alpha_1 \in \{\alpha_1^{(1)}, \dots, \alpha_1^{(d)}\}$, $(t, x, v) \in (0, \infty) \times \mathbf{R}^1 \times \{v_1, \dots, v_N\}$,

$$(1.3) \quad \lim_{\epsilon \downarrow 0} p_\epsilon(t/\epsilon, x - \alpha_1 t/\epsilon, v)$$

$$= \sum_{\{i: \alpha_1^{(i)} = \alpha_1\}} e_0^{(i)} \int_{-\infty}^{\infty} \frac{e^{-y^2/2}}{(2\pi)^{1/2}} \langle f, e_0^{(i)} \rangle (x + y(2\alpha_2^{(i)} t)^{1/2}) dy,$$

where $\alpha_2^{(i)}$ are positive real numbers given by formula (2.10).

In case the off-diagonal elements of Q are positive, the above results reduce to (respectively) the weak law of large numbers and the central limit theorem