

# *A Characterization of Entire Functions of Exponential Type and $M$ -Bounded Index*

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**1. Introduction.** An entire function of bounded index  $N$  is of exponential type at most  $N + 1$  [9] but an entire function of exponential type need not be of bounded index [9]. We define here functions of  $M$ -bounded index and show that a function of  $M$ -bounded index  $N$  is of exponential type at most  $N + 1$  and conversely a function of exponential type is always of  $M$ -bounded index.

An entire function  $f$  is said to be of bounded index if there exists an integer  $N$  such that

$$\max_{0 \leq i \leq N} \left\{ \frac{|f^{(i)}(z)|}{i!} \right\} \geq \frac{|f^{(n)}(z)|}{n!}$$

for all  $n$  and  $z$ . The least such integer  $N$  is called the index of  $f$  ([4], [6]).

Write  $M_n(r) = M(r, f^{(n)}) = \max_{|z|=r} \{|f^{(n)}(z)|\}$ , where  $f^{(0)}(z) = f(z)$ ,  $M_0(r) = M(r)$ .

An entire function  $f$  is said to be of  $M$ -bounded index if there exists an integer  $N$  such that

$$\max_{0 \leq i \leq N} \left\{ \frac{M_i(r)}{i!} \right\} \geq \frac{M_n(r)}{n!}$$

for all  $r \geq 0$  and all  $n$ . The least such integer  $N$  is called the  $M$ -index of  $f$ . It is easy to see that an entire function  $f$  of index  $N$  is also of  $M$ -index not exceeding  $N$ . The following example shows that the  $M$ -index, which we shall write as  $N(M)$ , may be less than the index  $N$ :

**Example.** For  $f(z) = ze^z$ ,  $N = 2$ ,  $N(M) = 1$ .

We will see later that there are functions of  $M$ -index which are not of bounded index.

Write  $T = \limsup_{r \rightarrow \infty} (\log M(r, f)/r)$ . Functions for which  $T < \infty$  are said to be of exponential type  $T$ . We state some properties of functions of  $M$ -bounded