

Operators With a Unique Positive Near-Approximant

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§1. Introduction. This paper continues the research began in [4]. Both papers deal with problems suggested by P. R. Halmos in [6]; these problems deal with approximating a given operator with a nonnegative operator. Professor Halmos asked for a characterization of when a given operator has a unique nonnegative operator closest to it.

In this paper we consider two different norms that might be used for measuring the “closeness” of operators. One is the usual operator norm and the other is a topologically equivalent norm which has some computational advantages. In the latter norm we give a complete solution to the above problem.

§2. Preliminaries. By operator we shall mean a bounded linear map of the complex Hilbert space H into itself. If S is a subspace of H then we denote the restriction of T to S by $T|S$; the spectrum of T is $\sigma(T)$; we simply write c when we mean cI with I the identity transformation. To say that T is normaloid means that the numerical radius of T , denoted $w(T)$, is equal to the norm of T , denoted $\|T\|$.

We shall follow Halmos' convention of using “positive operator” as synonymous with “nonnegative operator”; thus a positive operator is not necessarily invertible. We also use Halmos's phrase “positive approximant” to mean the best approximation in operator norm among the positive operators. For the reader's convenience we restate the following results proved by Halmos in [6].

2.1 Theorem. *If $B + iC$ is the usual Cartesian representation for the operator T , then*

$$\inf \{ \|T - P\| : P \geq 0 \} = \inf \{ r : B + (r^2 - C^2)^{1/2} \geq 0 \}.$$

The first infimum shall henceforth be denoted $\delta(T)$.