

Almost Every Proper Isometry is a Shift

LAWRENCE G. BROWN

Communicated by JOEL PINCUS

An operator S on a Hilbert space \mathcal{H} is called a shift if there is an orthonormal basis $\{e_n : n = 1, 2, \dots\}$ such that $Se_n = e_{n+1}$. More generally, an operator unitarily equivalent to the direct sum of α shifts is called a shift of multiplicity α , where α can be any positive cardinal number. It is known ([1]) that any isometry T of \mathcal{H} onto a proper subspace can be approximated in norm by a shift, whose multiplicity is necessarily the corank of T . We will show that the set of shifts is a G_δ , hence justifying the title, and derive some consequences.

If \mathcal{A} is a norm closed subset of bounded operators on the Hilbert space \mathcal{H} , we say almost every element of \mathcal{A} has property P if $\{T \in \mathcal{A} : T \text{ has property } P\}$ contains a dense G_δ of \mathcal{A} . Let \mathcal{U} denote the set of unitary operators on \mathcal{H} , $\mathcal{I}_\mathcal{L}$ the set of isometries of \mathcal{H} onto the proper subspace \mathcal{L} , and \mathcal{I} the set of proper isometries ($\mathcal{I} = \bigcup_{\mathcal{L}} \mathcal{I}_\mathcal{L}$). Also for T an isometry T_K denotes $\{T' : T' \text{ is an isometry and } T' - T \text{ is compact}\}$. Every element of T_K has the same corank as T .

Theorem 1. *If \mathcal{H} is a Hilbert space of countably infinite dimension, then the set of shifts is a G_δ subset of \mathcal{I} , and hence almost every element of \mathcal{I} and almost every element of T_K ($T \in \mathcal{I}$) is a shift.*

Proof. If $T \in \mathcal{I}$ and $\mathfrak{M} = T(\mathcal{H})^\perp$, then the spaces $T^n\mathfrak{M}$, $n = 0, 1, \dots$, are orthogonal and T is a shift if and only if they span \mathcal{H} . Now for $\epsilon > 0$ and $x \in \mathcal{H}$ let $S(\epsilon, x) = \{T \in \mathcal{I} : \|x - (I - TT^*)y_0 - \sum_{n=1}^k T^n(I - TT^*)y_n\| < \epsilon\}$, for some positive integer k and some $y_0, y_1, \dots, y_k \in \mathcal{H}$. $S(\epsilon, x)$ is obviously open. If $\{x_m : m = 1, 2, \dots\}$ is dense in \mathcal{H} , then the set of shifts is $\bigcap_{\epsilon, m=1}^\infty S(1/\epsilon, x_m)$. Since it was shown in [1] that shifts are dense in both \mathcal{I} and T_K , the theorem follows.

Remark. The original motivation for this was to find out whether the map $U \rightarrow U^{-1}SU$, from \mathcal{U} onto the set of shifts of fixed multiplicity, is open. An affirmative answer would have implied the theorem, but the answer is negative since S_K is separable whereas $\{U \in \mathcal{U} : U^{-1}SU \in S_K\}$ is not separable.

Corollary 1. *If $U_0 \in \mathcal{U}$ and $T \in \mathcal{I}$, then for almost every $U \in \mathcal{U}$ and for almost every $U \in U_{0K} UT$ is a shift.*