

Direct and Inverse Theorems for Multidimensional Spline Approximation

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1. Introduction. In two recent papers Scherer [9, 10] has exploited a general approximation theoretic result of Butzer and Scherer [2, 3] to obtain direct and inverse theorems as well as saturation results for spline approximation of continuous and of L_p functions.

The purpose of this paper is to apply the Butzer–Scherer result to obtain direct and inverse theorems as well as saturation theorems for multidimensional spline approximation of functions in certain Sobolev spaces. In order to employ the Butzer–Scherer Theorem, a Jackson-type estimate and a Bernstein-type inequality are obtained. For convenience, we restrict the discussion to the case of two dimensions; the extension to higher dimensions is straightforward.

2. Multidimensional Jackson-type estimates. In this section we give an estimate for how well functions in the Sobolev spaces $W^{k,p}(H)$, $1 \leq p \leq \infty$, can be approximated by splines, where H is the rectangle $H = [a, b] \times [c, d]$.

Let $m_1, n_1 \geq 0$ and $\Delta_1 = \{a = x_0 < x_1 < \cdots < x_{n_1+1} = b\}$. We define $S_{m_1, \Delta_1} = \{f \in H_p^{m_1}[a, b] : f^{(m_1+1)}(x) = 0 \text{ for } x \in (x_i, x_{i+1}), i = 0, 1, \dots, n_1\}$, where for $m_1 > 0$

$$H_p^{m_1}[a, b] = \{f \in C^{m_1-1}[a, b] : f^{(m_1-1)} \text{ is absolutely continuous, } f^{(m_1)} \in L_p[a, b]\},$$

and

$$H_p^0[a, b] = L_p[a, b].$$

For $m_2, n_2 \geq 0$ and $\Delta_2 = \{c = y_0 < y_1 < \cdots < y_{n_2+1} = d\}$, S_{m_2, Δ_2} is defined similarly. We are interested in the class of tensor product splines

$$S_{m_1, \Delta_1} \otimes S_{m_2, \Delta_2} = \{s(x, y) = s_1(x)s_2(y) : s_1 \in S_{m_1, \Delta_1}, s_2 \in S_{m_2, \Delta_2}\}$$