

Compact, Nuclear, and Hilbert-Schmidt Composition Operators On H^2

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Communicated by the Editors

Introduction. Let φ be an analytic function taking the open unit disc U into itself. It is well known [15, Theorem 1], [5, sec. 2.6, page 29] that the *composition operator* C_φ defined by the equation

$$C_\varphi f(z) = f(\varphi(z)) \quad (z \text{ in } U)$$

is a bounded linear operator on each of the Hardy spaces H^p ($0 < p < \infty$). In this paper we study the interplay between certain geometric properties of φ and functional analytic properties of C_φ ; emphasizing for simplicity the case where C_φ operates on the Hilbert space H^2 . For univalent maps φ our results express quantitatively the fact that C_φ will be a compact, and perhaps even a Hilbert-Schmidt or trace class (*i.e.* nuclear) operator if the boundary of $\varphi(U)$ touches the unit circle infrequently and sharply; while it will fail to be compact if the boundary of $\varphi(U)$ touches the unit circle too smoothly, even at a single point. For example, it is easy to see that C_φ is compact (in fact, in the trace class) if φ takes U into a disc $|z| \leq r < 1$. On the other hand, if $\varphi(z) = z$, then C_φ is the identity operator, which is not compact.

Much of our work generalizes to other H^p spaces, and we discuss these matters in the last section of the paper.

The first results of this type were obtained by H. J. Schwartz [17]. First Schwartz observed [17, Theorem 2.6, page 23] that the condition:

$$|\varphi(e^{it})| < 1 \text{ a.e.}$$

is necessary for the compactness of C_φ , but is not sufficient (here $\varphi(e^{it}) = \lim_{r \rightarrow 1-} \varphi(re^{it})$, where the limit exists *a.e.* by Fatou's radial limit theorem [5, Theorem 1.3]). In fact he showed that the function

$$(0.1) \quad \varphi(z) = (1 + z)/2,$$

which takes U onto a disc internally tangent to the unit circle, induces a non-compact composition operator, even though its boundary function has modulus