

Symmetric Singular Integral Operators

JOEL D. PINCUS

In this paper we study singular integral operators of the form

$$\mathfrak{L}x(\lambda) = A(\lambda)x(\lambda) + \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{\bar{k}(\lambda)k(\mu)}{\mu - \lambda} x(\mu) d\mu$$

when the coefficients $A(\lambda)$ and $k(\lambda)$ are permitted to be unbounded. We assume only that $A(\lambda)$ defines a self-adjoint multiplication operator, and that $\int_{-\infty}^{\infty} |k(\lambda)|^2 d\lambda < \infty$.

We give necessary and sufficient conditions for \mathfrak{L} , with a certain specified domain, to be maximally symmetric; and, when these conditions are satisfied, we compute the deficiency indices of \mathfrak{L} , give a complete spectral theory for the minimal self-adjoint dilation of \mathfrak{L} , and show how to decide when or if \mathfrak{L} has a self-adjoint restriction to one of its invariant subspaces.

Let

$$A_n(\lambda) = \begin{cases} A(\lambda) & \text{if } |A(\lambda)| \leq n \\ 0 & \text{otherwise} \end{cases}$$

$$k_n(\lambda) = \begin{cases} k(\lambda) & \text{if } |k(\lambda)| \leq n \\ 0 & \text{otherwise} \end{cases}$$

and set

$$L_n x(\lambda) = A_n x(\lambda) + \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{\bar{k}_n(\lambda)k_n(\mu)}{\mu - \lambda} x(\mu) d\mu.$$

L_n is a self-adjoint operator whose spectral theory has been completely elucidated in other papers. [1], [2].

It is known that the weak limit $w\text{-}\lim_{n \rightarrow \infty} (L_n - \ell)^{-1} = R(\ell)$ exists, as a bounded operator on $L_2(-\infty, \infty)$, whenever $\text{Im } \ell \neq 0$. [3].

Let $g_n(\nu, \lambda)$ be the characteristic function of the set of points (ν, λ) such that $A_n(\lambda) - |k_n(\lambda)|^2 < \nu < A_n(\lambda) + |k_n(\lambda)|^2$, $\lambda \in \text{supp. } k_n(\lambda)$. Let

$$E_n(\ell, z) = \exp \left(\frac{1}{2\pi i} \iint g_n(\nu, \mu) \frac{d\nu}{\nu - \ell} \frac{d\mu}{\mu - z} \right).$$