

Some Cyclic and Non-Cyclic Vectors of Certain Operators

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§1. Introduction and notation. The cyclic vectors of the backward unilateral shift on H^2 have been exhaustively studied by Douglas, Shapiro, and Shields in [1]. One of the many striking theorems they prove (Theorem 4.1.1) concerns the relation between cyclicity and rational approximation. They note that this theorem may be reformulated in terms of the kernel function k_λ of H^2 . The theorem then reads as follows:

Theorem 0. (*Tumarkin, Douglas, Shapiro, Shields*). A function $f \in H^2$ is non-cyclic if and only if $f = \lim f_n$ where $f_n \in \text{span} \{k_{\lambda_1 n}, \dots, k_{\lambda_n n}\}$ and $\sum_i (1 - |\lambda_i^n|) \leq \alpha, n = 1, 2, \dots$.

The authors of [1] ask about possible generalizations of this theorem to other operators on other spaces. The main purpose of this paper is to make some observations on this question. In §2 we show that the “if” part of *Theorem 0* generalizes to a class of Banach space operators including certain Hilbert space contractions. In §3 we make some observations about operators having “many” eigenvalues on or near the unit circle. In §4 we show that the “only if” part of *Theorem 0* fails for a large class of weighted shifts (or rather for their adjoints). As a by-product we are able to exhibit a class of weighted shifts having invariant subspaces which are not commuting ranges. Ralph Gellar has informed us that he has also produced this phenomenon, but by quite different methods. Finally we discuss briefly a special kind of cyclic vector for the backward shift.

X will denote a complex Banach space, H a complex Hilbert space and $L(X)$ the class of (bounded) linear operators on X . If $T \in L(X)$ and $f \in X$, then $Z(f)$ denotes the (closed) cyclic subspace for T generated by f . The open unit disc in the complex plane \mathbf{C} will be denoted by Δ . The determinant function is denoted by \det .

A $T \in L(X)$ is said to be polynomially bounded with constant ξ if $\|p(T)\| \leq \xi \|p\|_\infty$ for each polynomial p where $\|p\|_\infty = \sup \{|p(z)| \mid z \in \Delta\}$. This inequality