

On Reductive Operators

ALI A. JAFARIAN

Communicated by the Editors

Introduction. Let \mathcal{H} be a Hilbert space and $\mathcal{B}(\mathcal{H})$ the algebra of all bounded linear operators on \mathcal{H} . An operator $T \in \mathcal{B}(\mathcal{H})$ is *reductive* if every invariant subspace M of T reduces T ; i.e., $TM \subset M$ implies $TM^\perp \subset M^\perp$. Is every reductive operator normal? This is the reductive operator problem, which is a well-known unsolved problem. It was shown by J. Dyer and P. Porcelli [8] that the reductive operator problem is equivalent to the invariant subspace problem (on a Hilbert space). Until recently there were very few known results on this problem; (compact [1] and polynomially compact ([13], [14]) reductive operators were known to be normal). Some results have recently been obtained. E. Nordgren, H. Radjavi and P. Rosenthal [11] proved that a reductive operator T whose spectrum lies in a C^2 -Jordan curve J and whose resolvent satisfies the growth condition $\|R(\lambda; T)\| \leq \exp \{K[\text{dist}(\lambda, J)]^{-m}\}$, for $\lambda \notin J$, where $K > 0$ and m is a positive integer, is spectral. If moreover $TT^* - T^*T$ is compact, then T is normal. M. Radjabalipour [12] showed that if $\sigma(T)$ lies in a C^1 -Jordan curve J and if $\|R(\lambda; T)\| \leq K[\text{dist}(\lambda, J)]^{-m}$, for $\lambda \notin J$, where $K > 0$ and $m > 0$ is an integer, then reductivity of T implies normality. In this paper we generalize the above results of [11] and [12].

§0. Preliminaries and notation. Let \mathbf{C} denote the complex plane. An operator $T \in \mathcal{B}(\mathcal{H})$ has the *single-valued extension property* ([4], [5]) if whenever $f: \Omega \rightarrow \mathcal{H}$, Ω an open subset of \mathbf{C} , is an analytic function such that $(\lambda - T)f(\lambda) \equiv 0$ on Ω , then it follows that $f(\lambda) \equiv 0$ on Ω . Let $h \in \mathcal{H}$ and $T \in \mathcal{B}(\mathcal{H})$ have the single-valued extension property. The *local resolvent of T at h* , $\rho_T(h)$, is the set of all $\lambda_0 \in \mathbf{C}$ such that there is an analytic \mathcal{H} -valued function u defined on a neighborhood V of λ_0 with the property $(\lambda - T)u(\lambda) \equiv h$, on V . If T has the single-valued extension property, then there exists a unique analytic function $h_T(\lambda)$ defined on $\rho_T(h)$ and satisfying $(\lambda - T)h_T(\lambda) \equiv h$, on $\rho_T(h)$.

It is clear that $\rho_T(h)$ is an open set and $\rho(T) \subset \rho_T(h)$. The *local spectrum of T at h* , is defined to be $\sigma_T(h) = \mathbf{C} \setminus \rho_T(h)$. Then $\sigma_T(h)$ is a closed subset of $\sigma(T)$. If $F \subset \mathbf{C}$, define $\mathcal{H}_T(F) = \{h \in \mathcal{H}: \sigma_T(h) \subset F\}$. Clearly $\mathcal{H}_T(F) = \mathcal{H}_T(F \cap \sigma(T))$. It can be seen ([4]) that $\mathcal{H}_T(F)$ is a linear manifold; that if $F \subset \mathbf{C}$ is closed