

# On Semi-Coercive Nonlinear Problems

PETER HESS

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**Introduction.** In this note we investigate the solvability of nonlinear equations and inequalities in case the usual coerciveness assumption (which, in some sense, guarantees that the problem is *always* solvable) is replaced by the weaker condition of semi-coerciveness, *i.e.* coerciveness only with respect to a seminorm. Problems of this kind were previously considered by Fichera [4], Lions–Stampacchia [7], Browder [3] and Schatzman [8]. Their approach is essentially based on a direct decomposition of the basic space involved, and on a projection argument. This procedure is rather complicated and unnatural, especially for nonlinear operators. It is our aim to present a simple and transparent method which, roughly speaking, shows that semi-coercive problems are solved only in cases in which they are actually coercive ones.

The abstract results are given in Section 1. Section 2 contains two new applications to nonlinear elliptic boundary value problems.

**Section 1.** Let  $V$  be a real reflexive Banach space with norm  $\|\cdot\|_V$  and dual  $V^*$ , and let  $(f, u)$  denote the duality pairing between  $f \in V^*$  and  $u \in V$ . Strong and weak convergence are expressed by the symbols “ $\rightarrow$ ” and “ $\rightharpoonup$ ”, respectively.

**Definition.** *The mapping  $T : V \rightarrow V^*$  is pseudo-monotone provided*

(i)  *$T$  is continuous from finite-dimensional subspaces of  $V$  to the weak topology on  $V^*$ ;*

(ii) *if  $\{u_n\}$  is a sequence in  $V$  such that  $u_n \rightharpoonup u$  and  $\limsup (Tu_n, u_n - u) \leq 0$ , then*

$$(Tu, u - v) \leq \liminf (Tu_n, u_n - v) \text{ for all } v \in V.$$

Let further  $\varphi : V \rightarrow (-\infty, +\infty]$  be a convex, lower semicontinuous function with  $\varphi(0) = 0$ . For given  $f \in V^*$  we are concerned with the problem of finding  $u \in V$  such that

$$(1.1) \quad (Tu - f, v - u) + \varphi(v) - \varphi(u) \geq 0 \text{ for all } v \in V.$$

A criterion for solvability of (1.1) is given by the following