

# Remark on Tonelli's Theorem on Integration in Product Spaces—II

A. MUKHERJEA

*Communicated by the Editors*

Let  $(X, U, \beta_1)$  and  $(Y, V, \beta_2)$  be any two arbitrary measure spaces where  $U$  and  $V$  are sigma-algebras of subsets of  $X$  and  $Y$  respectively, and  $\beta_1$  and  $\beta_2$  are non-negative non-zero (possibly infinite) measures on  $U$  and  $V$  respectively. Let  $U \times V$  be the sigma-algebra generated by the measurable rectangles. Let  $\beta^*$  be the outer measure induced on the subsets of  $X \times Y$  by the measure  $\beta$  defined on the algebra of finite disjoint union of measurable rectangles by  $\beta(\cup P_i \times Q_i) = \sum \beta_1(P_i)\beta_2(Q_i)$ . Then  $\beta^*$  restricted on the class of  $\beta^*$ -measurable sets (see [6], p. 251) containing  $U \times V$  as a subset, is a measure that is complete (that is, every subset of a measurable set of measure zero is measurable). The measure  $\beta^*$  restricted on  $U \times V$  will be called the product of  $\beta_1$  and  $\beta_2$  and denoted by  $\beta_1 \times \beta_2$ . The measure  $\beta_1$  is called semi-finite if every set in  $U$  with infinite  $\beta_1$ -measure has a subset in  $U$  with finite positive measure. Every sigma-finite measure is semifinite, but not conversely.

Counting measure on an uncountable set is a semifinite measure, that is not sigma-finite. One can construct semi-finite non-sigma-finite non-atomic (that is, containing no atom see p. 321 in (6)) measures in the following way.

Let  $(X_p, U_p, \beta_p)$  be an uncountable family of pairwise disjoint non-atomic probability measure spaces. Let  $X_0 = \cup X_p$  and  $U_0 = \{A \subset X_0 : \text{either } A \text{ or } A^c \text{ intersects at most countably many } X_p \text{ and } A \cap X_p \in U_p\}$ . If  $A$  in  $U_0$  intersects at most countably many  $X_p$ , we define  $\beta_0(A) = \sum \beta_p(A \cap X_p)$ ; otherwise, we define  $\beta_0(A) = \infty$ . It follows easily that  $(X_0, U_0, \beta_0)$  is a semi-finite non-sigma-finite non-atomic measure space.

One can also consider an important class of measures called Hausdorff measures which provide interesting examples of non-atomic non-sigma-finite, but semi-finite measures. Let  $Z$  be an uncountable complete separable ultra-metric space or a  $n$ -dimensional Euclidean space. By Theorem 35, p. 63 in [5], there is a monotonic increasing, continuous (from the right) function  $g(t)$  with  $g(0) = 0$  such that if  $\beta^g$  is the Hausdorff measure induced by  $g$  on  $Z$ , then  $0 < \beta^g(C) < \infty$  for some compact set  $C$ . If we define  $h^2(t) = g(t)$  for  $t \geq 0$ , then  $\lim g(t)/h(t) = 0$