

Saturated Chains in Noetherian Rings

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Introduction. Let R be a domain and x be an indeterminate. If P is a prime in R and Q is a prime in $R[x]$ such that $Q \cap R = P$ but $Q \neq PR[x]$, we will call Q an upper to P . We prove the following result, which combines and extends Propositions 2.2 and 4.7 of [5].

Proposition. *If (R, M) is a local domain and $n > 1$ then the following are equivalent: (i) There is a rank 1 prime of R with corank $n - 1$. (ii) There are infinitely many rank 1 primes of R with corank $n - 1$. (iii) There is an upper K to O in $R[x]$ such that $K \subset MR[x]$ and $\text{rank}(MR[x]/K) = n - 1$. (iv) There is an upper K to O and an upper Q to M such that $K \subset Q$ and $\text{rank}(Q/K) = n$. We also prove that if P is a rank n prime in a Noetherian ring, then there are only finitely many primes Q satisfying $P \subset Q$, $\text{rank}(Q/P) = 1$ and $\text{rank } Q > n + 1$.*

We deduce several consequences of these two results. For instance if P is a prime in a Noetherian domain R , and if in $R[x]$ there is a saturated chain of primes of length n from O to $PR[x]$ then in R there is a saturated chain of primes of length n from O to P .

Throughout this paper we have depended heavily on the work done in [5], both as a guide to overall goals, as well as a source of useful techniques. The author is indebted to Professor Ratliff for that and for personal encouragement.

Terminology. All rings are commutative with identity. We will say that the prime P' is *directly above* the prime P if $P \subset P'$ and $\text{rank}(P'/P) = 1$. We will say that P' is *directly between* P and P'' if $P \subset P' \subset P''$ and $\text{rank}(P'/P) = 1 = \text{rank}(P''/P')$. Finally the phrase "*almost all*" will mean "all but finitely many". All other definitions are as in [3].

Our first theorem was clearly foreshadowed by [6, Lemma 2.1].

Theorem 1. *Let P be a rank n prime in a Noetherian ring. Then almost all the primes directly above P have rank $n + 1$.*

Proof. Let I be generated by n elements with P minimal over I [3, Theorem 153]. Let $P = P_1, P_2, \dots, P_m$ be all primes minimal over I . Suppose that the