

Norm Denseness of Reducible Hyponormal Operators

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1. Only bounded operators T on a separable infinite-dimensional Hilbert space will be considered. The question as to whether every T is the norm limit of reducible operators was raised by Halmos [1], p. 919, and apparently is still open. It may be noted however that operators having a non-trivial invariant subspace are norm dense; in fact, even operators with disconnected spectra are dense (Herrero and Salinas [2]). It was shown in [1], p. 922, that every isometry can be (uniformly) approximated by reducible operators, even by ones in the class R_1 of operators with reducing eigenvectors. Further, the same argument shows that, more generally, if T is hyponormal, that is, if

$$(1.1) \quad T^*T - TT^* \geq 0,$$

then T is also in the closure of R_1 . In fact, what is needed is that T and T^* have common approximate eigenvectors, that is, for every $\delta > 0$, there exists a complex number t and a unit vector $x = x(t, \delta)$ for which both

$$(1.2) \quad \|(T - tI)x\| < \delta \quad \text{and} \quad \|(T^* - \bar{t}I)x\| < \delta.$$

For $z \notin sp(T)$, let $R(z) = (T - zI)^{-1}$ and $d(z) = \inf \{|z - s| : s \in sp(T)\}$. If t is a point of the boundary of $sp(T)$ and if $d(z) \|R(z)\| \rightarrow 1$ as $z \rightarrow s$ ($z \notin sp(T)$) holds for all s lying both in the boundary of $sp(T)$ and in some neighborhood of t , then (1.2) holds (see Putnam [4], Th. 2, pp. 506-507). Thus, such a T also lies in the closure of R_1 . It is known that a hyponormal operator T even satisfies $d(z) \|R(z)\| \equiv 1$, $z \notin sp(T)$ (see Stampfli [6], [7]).

It will be shown below that every hyponormal operator can be uniformly approximated by reducible *hyponormal* operators (not necessarily by ones of the class R_1 however). Thus, there will be proved the following

Theorem. *If T is hyponormal then there exists a sequence of reducible hyponormal operators T_1, T_2, \dots for which $\|T - T_n\| \rightarrow 0$ as $n \rightarrow \infty$.*

2. It will be convenient to isolate the following fact as a simple