

The Smoothness of Solutions to Nonlinear Variational Inequalities

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In this paper we propose to study several questions about the smoothness of solutions to non-linear variational inequalities with obstacles. We shall describe our conclusions, providing a brief account of their origins. Let Ω be a bounded open connected subset of \mathbf{R}^n with smooth boundary $\partial\Omega$. Let $a(p) = (a_1(p), \dots, a_n(p))$, $p = (p_1, \dots, p_n) \in \mathbf{R}^n$, be a C^2 vector field which satisfies the condition:

$$(1.1) \quad \text{For each compact } C \subset \mathbf{R}^n, \text{ there exists a } \nu = \nu(C) > 0 \text{ such that} \\ (a(p) - a(q))(p - q) \geq \nu |p - q|^2 \text{ for all } p, q \in C.$$

Let $\psi \in C^2(\bar{\Omega})$ satisfy $\psi \leq 0$ on $\partial\Omega$ and let $f \in C^1(\bar{\Omega})$. With $\mathbf{K} = \mathbf{K}_\psi$ the convex set of lipschitz functions v which satisfy $v \geq \psi$ in Ω and $v = 0$ on $\partial\Omega$, we consider the variational inequality

$$(1.2) \quad u \in \mathbf{K}: \int_{\Omega} a_i(Du) D_i(v - u) dx \geq \int_{\Omega} f(v - u) dx \quad \text{for all } v \in \mathbf{K}.$$

Our first object is to prove that the second derivatives of a solution u to (1.2), $D_{ik}u$, are locally bounded in Ω , or in other words, that $u \in C^{1,1}(\Omega)$. Results in this direction were first attained by one of the authors in the special cases $n = 2$, ψ concave, $f = 0$, and $a_i(p) = p_i$ or $p_i/(1 + |p|^2)^{1/2}$ where the conclusion was necessary to decide the smoothness of the curve ∂I . Here $I = \{x \in \Omega: u(x) = \psi(x)\}$ is the coincidence set of the solution. We refer to ([7], [8], or [9]). Very recently and by a different technique, J. Frehse has published a proof of the boundedness of the second derivatives in the case $a_i(p) = p_i$, [5].

The method by which we shall approach this problem has been studied already by one of the authors in ([1], [2]). We consider the variational inequality as a multivalued equation involving a monotone graph. In this paper we make a new observation crucial for our proof: that the particular monotone graph with which we deal is also *concave*, leads us, by differentiating the original equation twice, to uniform bounds for second derivatives.