

Local Invariants of the Riemannian Metric for Two Dimensional Manifolds

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Introduction. In this paper we discuss the invariant maps from Riemannian metrics on two dimensional manifolds to forms which are given by a local formula in the derivatives of the metric. For such a map P , we let $\text{ord}(P)$ denote the total number of differentiations involved in the formula. The scalar curvature K is such a map from metrics to functions which is of order 2. We can construct other invariant maps from K by using addition, exterior multiplication, exterior differentiation (d), and the Hodge $*$ operator. In section 1, we show that all invariant maps of the metric which are given by local formulas are obtained from K in this way. This result, of course, is false in dimension ≥ 3 .

In section 2, we prove an analogue of the Poincare lemma which we state as follows: suppose that P is an invariant map from metrics to 1-forms such that P is always a closed 1-form. Then there is a map Q from metrics to functions such that $dQ = P$. This implies that every functorial closed form is functorially exact and that there are no functorial maps from metrics to $H^1(M; R)$ except the zero map.

We assume that the manifold M is oriented in sections 1 through 3; in section 4 we study unoriented manifolds. The assumption that M is oriented permits us to identify

$$\Lambda^2(T^*M) = \Lambda^0(T^*M).$$

In section 3, we prove an analogue of the Hodge decomposition theorem. The classical Hodge decomposition theorem states that if P is any function such that $\int P = 0$, then there is a function Q such that $DQ = P$. $D = d * d$ is the usual Laplace operator. This theorem has the formal analogue which we state as follows: let P be a map from metrics to 2-forms (or functions) which is of order $n > 2$. We are excluding the scalar curvature K as a possibility for P by assuming that $n > 2$. Suppose that

$$\int_{T_x} P(\text{metric}) = 0$$