

Maximal Smoothing Operators

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Introduction. In this paper we study the differential properties of functions in the class $L^p_\alpha(\mathbf{R}^n)$ (see below for definition) where $0 < \alpha < n$ and $n/\alpha < p < \infty$. Our main concern is to measure the set of points at which pointwise differentials exist for functions in this class. We consider two problems, the differential of highest order and the lower order differentials. We first show (Corollaries of Theorem I) that if $f \in L^p_\alpha$ and $k = [\alpha]$, the integral part of α , then f has a k -differential almost everywhere. Moreover, if $\alpha - k > 0$, the $k + 1$ order difference of f at x and size h tends to zero as $|h| \rightarrow 0$ faster than $|h|^\alpha$ (for a.e. x). This result for $\alpha = 1$ is due to A. P. Calderon [4] and for integral $\alpha > 1$ is due to J. Rešetnjah [7]. See also [9].

Concerning a lower order differential, say of order β , we prove in Theorem II that if $f \in L^p_\alpha$, then the β -differential of f exists except for a set E which for each $\epsilon > 0$ satisfies $H_{n-(\alpha-\beta)p+\epsilon}(E) = 0$. Here $H_\mu(E)$ denotes the Hausdorff measure of E of order μ .

The notation throughout this paper is standard and the meaning should be clear in context. The space of functions considered in this work is the class $L^p_\alpha(\mathbf{R}^n)$, $\alpha > 0$. This is the set of functions f defined on \mathbf{R}^n for which there exists $g \in L^p(\mathbf{R}^n)$ such that

$$f(x) = G_\alpha * g(x) = \int_{\mathbf{R}^n} G_\alpha(x-y)g(y) dy$$

where $G_\alpha(x)$ is the Bessel kernel of order α . (The Fourier transform of $G_\alpha(x)$ is $(1 + |x|^2)^{-\alpha/2}$. See [10]). The L^p_α norm of f , denoted by $\|f\|_{L^p_\alpha(\mathbf{R}^n)}$, is defined to be the L^p norm of g , denoted by $\|g\|_{L^p(\mathbf{R}^n)}$.

If h and $x \in \mathbf{R}^n$ we define $\Delta_h(f)(x) = f(x+h) - f(x)$, and for k a positive integer we define

$$\Delta_h^k f(x) = \Delta_h(\Delta_h^{k-1}f)(x).$$

Theorem I. Suppose $f \in L^p_\alpha(\mathbf{R}^n)$ with $n/\alpha < p < \infty$, $0 < \alpha < n$, and let k be the first integer $\geq \alpha$ ($k = [\alpha + 1] - 1$). The function,