

# *The Measure of Non-Compactness in $L^p$ for a Class of Integral Operators*

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Let  $(a, b)$  be any interval of reals,  $-\infty \leq a < b \leq \infty$ . Denote by  $L^p (= L^p(a, b))$ ,  $1 \leq p \leq \infty$ , the Lebesgue space of complex valued functions over  $(a, b)$ . For complex valued measurable functions  $u$  and  $v$  we study here operators defined in  $L^p$  by the operation

$$(0.1) \quad f(y) \rightarrow u(x) \int_a^x v(y)f(y) dy$$

where the part of the support of  $u(v)$  in any neighbourhood of  $b(a)$  has non-zero measure. The results provide also properties for operators defined by the formal dual operation with respect to the usual Banach space pairing. Such operators as these arise naturally in the study of certain problems in ordinary differential equations, for example.

It is known that the operation (0.1) defines a bounded operator  $T$  in  $L^p$  if and only if  $u \in L^p(c, b)$  and  $v \in L^q(a, c)$  for every  $c$ ,  $a < c < b$ ,  $1/p + 1/q = 1$ , and the function

$$(0.2) \quad (a, b) \ni x \mapsto k(x) = \left( \int_x^b |u(y)|^p dy \right)^{1/p} \left( \int_a^x |v(y)|^q dy \right)^{1/q}$$

is bounded. The usual interpretations are to be made when  $p = 1$ , or  $p = \infty$ : this is to be done throughout, whenever necessary. In addition, letting

$$(0.3) \quad K = \sup_{a < x < b} k(x),$$

the norm of  $T$ ,  $\|T\|$ , satisfies

$$(0.4) \quad K \leq \|T\| \leq p^{1/p} q^{1/q} K.$$

The inequalities in (0.4) are best possible. This was established by G. Tomaselli in [9]; but see also [1], [4], [8], and [2], in part, for a special case.

Supposing the operation (0.1) defines a bounded operator  $T$  in  $L^p$  we present