

On Gaussian Sums over $GL(2, \mathbf{Z}/p^n)$

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Communicated by the Editors

Kondo [1] studied the Gaussian sum

$$W(A) = \sum_g R(g) \exp(2\pi i \operatorname{trace}(Ag)/p)$$

where R is an irreducible representation of $GL(2, \mathbf{Z}/p)$, p is prime, A is in $GL(2, \mathbf{Z}/p)$, and g is summed over $GL(2, \mathbf{Z}/p)$, and found that $W(A) = W(I)R(A)^{-1}$, and that $W(I)$ is always a nonzero scalar matrix. He actually proved his result for an arbitrary finite field, but I intend to go in the other direction, namely, modulo p^n . If R is thus an irreducible representation by complex matrices of $GL(2, \mathbf{Z}/p^n)$ and A is in $M(2, \mathbf{Z}/p^n)$, define the Gaussian sum $W(A)$ to be

$$W(A) = \sum_g R(g) \exp(2\pi i \operatorname{trace}(Ag)/p^n).$$

When $n > 1$, $W(I)$ can vanish. Nevertheless, $W(A)$ still behaves somewhat like a scalar multiple of $R(A)^{-1}$, even when A is singular, in the sense that we will be able to prove identities of the type $W(A)W(B) = cW(BA)$, at least for a certain class of representations. It is my ultimate intention to show how these identities lead to identities for the Hecke operators of type R defined by Shimura [2].

1. Although $W(A)$ depends on A modulo p^n , it is convenient to regard A as lying in $M(2, \mathbf{Z})$. As far as an identity of the type $W(A)W(B) = cW(BA)$ is concerned, we may as well suppose that A and B have p -power determinants, for we can write $A = A_1S$, $B = TB_1$, where A_1 and B_1 have p -power determinants, S and T have determinants not divisible by p , thus determining elements of $GL(2, \mathbf{Z}/p^n)$. Then $W(A)W(B) = R(S)^{-1}W(A_1)W(B_1)R(T)^{-1}$, while $W(BA) = R(S)^{-1}W(B_1A_1)R(T)^{-1}$, so $W(A)W(B) = cW(BA)$ if and only if $W(A_1)W(B_1) = cW(B_1A_1)$.

If $\Gamma = SL(2, \mathbf{Z})$, and A is in $M(2, \mathbf{Z})$, then the set $\Gamma A \Gamma$ splits into a finite disjoint union $\cup \Gamma A_i$ of right cosets, where the coset representatives A_i lie