

An Exponential Formula for Determining Functions

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Let T be a bounded completely nonnormal operator acting on a Hilbert space \mathfrak{H} for which the self-commutator $[T, T^*] = TT^* - T^*T \equiv D$ is trace class. The study of the unital C^* -algebra generated by T has led to the introduction of an operator valued function $E(\ell, z)$ defined on an auxillary ℓ_2 space h , called the determining function of T . This function is a complete unitary invariant for T and is quite helpful in studying the operator T (see the Remarks below) from both an analytic and geometric standpoint. The purpose of the present paper is to establish a basic property about the determining function: there exists a summable function $G(\cdot, \cdot)$ such that

$$(I) \quad \det E(\ell, z) = \exp \left\{ \frac{1}{2\pi i} \iint G(\nu, \mu) \frac{d\nu}{\nu - \ell} \frac{d\mu}{\mu - z} \right\}.$$

The function $G(\cdot, \cdot)$, called the principal function of T , is known to yield much information about the operator T (see [9]); however, proofs of its existence have appeared only in cases where the commutator was either nonpositive or nonnegative, or else satisfied some smoothness condition ([3]; [4]). Here we offer a version which holds in general.

The determining function. Let U and V denote the real and imaginary parts of T so that $T = U + iV$ is the usual Cartesian decomposition. If we write $C = (\pi/2)D$, then it follows easily that

$$[V, U] = VU - UV = \frac{1}{\pi i} C.$$

This identity together with the following construction will be the preparation we need in order to define the determining function.

Let the Schmidt series of C be given as

$$C = \sum (\text{Sgn } \lambda_n) \lambda_n^2 \phi_n(\cdot, \phi_n)$$

where the n^{th} eigenvalue of C is $(\text{Sgn } \lambda_n) \lambda_n^2$ and $\{\phi_n\}$ denotes the set of orthonormal eigenvectors of C completed if necessary. Let $\{\theta_n\}$ denote a fixed complete orthonormal set in an auxillary ℓ_2 space, h .