

# Saturation Theorems Related to the Mean Ergodic Theorem

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**1. Introduction.** Let  $X$  be a Banach space and  $T$  denote a bounded operator from  $X$  into  $X$ . The mean ergodic theorem on the behavior of the sequence of arithmetical means of the iterates  $T^n$  of a power bounded  $T$  (i.e.  $\|T^n\| \leq M$ ,  $n = 0, 1, 2, \dots$  for some  $M$ ) has been extended and generalized in many directions. Among these is a theorem due to Yosida [4] who replaced the power-bounded operator by an operator  $T$  with  $\|T^n\| = o(n)$  as  $n \rightarrow \infty$ . Observing that the arithmetical means are the Cesàro  $(C, 1)$  transform of the iterates  $T^n$  the following has been shown recently by Ramanujan and the author [3].

**Theorem A.** *Let  $X$  be a Banach space and  $T$  a bounded operator from  $X$  into  $X$ . Let  $(a_{nk})$  be a regular summability matrix and  $\{\gamma_n\}$  an increasing sequence of non-negative reals. Assume that*

- (1)  $T_n x = \sum_{k=0}^{\infty} a_{nk} T^k x$  exists for each  $x \in X$  and each  $n$ , and that  $\{T_n x\}$  ( $n \geq 0$ ) is weakly relatively compact;
- (2)  $\sum_{k=0}^{\infty} |a_{nk} - a_{n,k+1}| \gamma_{k+1} \rightarrow 0$ , as  $n \rightarrow \infty$ ;
- (3)  $\|T^n\| \leq \gamma_n$ .

*Then for each  $x \in X$ ,  $T_n x \rightarrow Px$  as  $n \rightarrow \infty$ , where  $P$  is the projection of  $X$  onto  $N$ , the null space of  $I - T$ , parallel to  $\bar{R}$ , the closure of the range of  $I - T$ . Moreover  $X = N \oplus \bar{R}$ .*

If  $T$  is power bounded and  $X$  is reflexive, then (1) is automatically satisfied. If the matrix  $(a_{nk})$  is the Cesàro matrix  $(C, r)$  ( $r \geq 1$ ) or the Abel "matrix", then (see [3] § 4)  $\{\gamma_n\}$  may be taken as any sequence satisfying  $\gamma_n = o(n)$  as  $n \rightarrow \infty$ ; and this reduces to Yosida's result [4] for  $(C, 1)$ .

Recently Butzer and Westphal [1] considered the question of how rapidly the  $(C, 1)$ —means of a power-bounded operator  $T$  on a reflexive space  $X$  converge to  $P$  in the strong sense; and also how well the Abel means of a power