

Morse Theory and Cell Decomposition on Kähler Manifolds and Related Spaces

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In [6], by using a technique due to Frankel [4], we showed how the Kähler homogeneous structure of the spaces $G/C(T')$ may be exploited to give a non-degenerate Morse function which allows one to read $H_*(G/C(T'); \mathbb{Z})$ from the diagram of the compact semisimple Lie group G . Here we extend Frankel's technique to certain (not necessarily Kähler) submanifolds of Kähler manifolds. As a particular example, we shall compute the \mathbb{Z}_2 homology of the real flag manifolds.

We shall also discuss the question of cell decompositions of the spaces in question. In particular we shall show that two cell decompositions of the complex (and real) Grassmann manifolds due to Borel [1] and Ehresmann [3] are actually the same. This will be done via a cell decomposition coming from a Morse function.

We assume the reader has some knowledge of Lie theory, roots and the like.

§1. The technique of Frankel. Let M be a compact Kähler manifold with Kähler two-form Ω . Let $\Psi = \{\psi_t\}$ be a one-parameter group of isometries of M with fixed set F and Killing field (infinitesimal isometry) X . Then

Theorem. (Frankel [4].) *If either $F \neq \emptyset$ or the first real Betti number of M vanishes, then $\iota_X \Omega = -df$, where ι_X denotes interior product by X and f is a smooth function. The critical set of f is a finite union of disjoint nondegenerate critical manifolds in the sense of Bott [2] and coincides with F . If $x \in F$, the tangent space to M at x splits orthogonally as $TM_x = TF_x \oplus (\oplus e_r)$, where each e_r is two dimensional and invariant under both ψ_{t*} and the almost complex structure operator J . The index of f at x is twice the number of e_r that ψ_{t*} rotates negatively with respect to the orientation of J . Finally, the Morse-Bott inequalities are equalities:*

$$b_i(M; \mathbb{K}) = \sum b_{i-\lambda_\alpha}(F_\alpha; \mathbb{K})$$