

# *The Distant Future*

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**§1. Introduction.** Recall that via a Hilbert space isomorphism each stationary stochastic process, in the wide sense and without any continuity assumption, may be realized by selecting a suitable finite positive measure  $\mu$  on the Bohr group  $G$  (the group dual to the real line  $\mathbf{R}$  with the discrete topology) and letting the random variables be the characters  $\{\chi_\lambda\}_{\lambda \in \mathbf{R}}$  of  $G$  regarded as functions in  $L^2(\mu)$ . The *distant future* of the process is the intersection  $\bigcap_{\tau \in \mathbf{R}} \mathfrak{M}_\tau$ , where  $\mathfrak{M}_\tau$  is the closed linear span of the random variables  $\chi_\lambda$  with  $\lambda > \tau$ . In this paper we investigate the prediction problem of deciding in terms of  $\mu$  when the distant future consists of the zero function alone. If the process is continuous, then well-known criteria are given by the classic results of Szegő [14] and Kolmogorov [8]. On the other hand, if  $\mu$  is absolutely continuous with respect to Haar measure on  $G$ , then criteria may be found in the famous paper of Helson and Lowdenslager [7]. In the present paper, we solve the prediction problem without any supplementary hypothesis whatsoever.

To understand our solution, recall that  $\mathbf{R}$ , with the usual topology now, may be continuously imbedded in  $G$  as a dense one-parameter subgroup  $\{e_t\}_{t \in \mathbf{R}}$  and, therefore, we may regard  $\mathbf{R}$  as acting on  $G$  as a topological transformation group. The measure  $\mu$  on  $G$  is called quasi-invariant in case its class of null sets is preserved by this action. If  $\mu$  is quasi-invariant, then  $\rho(t, x)$  will denote the derivative  $d\mu_t/d\mu(x)$  where  $\mu_t(M) = \mu(M - e_t)$  ( $M$  a Borel set). Our solution to the prediction problem is this: The distant future of the process consists of the zero function alone if and only if 1)  $\mu$  is quasi-invariant and 2) there is a  $\mu$ -null set  $N$  such that for each  $x \notin N$ ,  $\log \rho(t, x)$ , as a function of  $t$ , belongs to  $L^1(dt/1 + t^2)$ .

Observe that the process is continuous precisely when  $\mu$  is concentrated on the subgroup  $\{e_t\}_{t \in \mathbf{R}}$  and, in this case,  $\mu$  is quasi-invariant precisely when it is equivalent to linear Lebesgue measure  $dt$  on  $\mathbf{R}$  transplanted to  $\{e_t\}_{t \in \mathbf{R}}$ . If  $w = d\mu/dt$  is regarded as a function on  $\mathbf{R}$ , then  $\rho(t, x) = w(s - t)/w(s)$ , if  $x = e_s$  and is zero otherwise. Hence the distant future vanishes when and only when  $\log w(t)$  lies in  $L^1(dt/1 + t^2)$ . This is the result of Szegő as extended by