

Infinitesimal Convexity Implies Local Convexity

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Communicated by the Editors

Suppose that M is a Riemannian manifold and that L is an isometrically imbedded, closed, oriented hypersurface. We say that L is *infinitesimally convex* if the second fundamental form of L is positive semidefinite at every point. L is *locally convex* if for every $p \in L$ there is a neighborhood U of $0 \in L_p$ (the tangent space at p) such that $\exp U$ has no points of M on the "inside" of L . By the *inside* of L we mean the component of a tubular neighborhood with L deleted which corresponds to the negative multiples of the orienting normal vector field under \exp . The exponential maps are those of M .

It is an immediate consequence of one of the standard interpretations of the second fundamental form, as the Hessian of a normal coordinate function restricted to L [BC, p. 198], that local convexity implies infinitesimal convexity. When M has constant curvature, the converse has been proved by doCarmo and Warner [CW]. It was doCarmo who brought my attention to the fact that the general case is not as obvious as it sounds.

The differentiability requirements for our proof of the converse seem to be C^4 , since at one step a curvature estimate is used to obtain a uniform bound for some focal distances in a family of surfaces. This bound probably could be obtained more directly in terms of metric components, and hence the differentiability requirements would be less.

The theorem is local, so we may assume that M is a geodesically convex neighborhood of p and that the exponential map of the normal bundle of L is a diffeomorphism from some neighborhood of the 0-section to M . Furthermore, we may assume that the sectional curvatures are bounded above.

The idea of the proof is to reduce it to the case where $\dim M = 2$, by constructing surfaces which sweep out a neighborhood of p and such that if $\exp U$ were to cross to the inside of L , then it would have to do so within one of these surfaces. In order to succeed with this approach we must insure that the intersection of a surface with L inherits the infinitesimal convexity of L and that the geodesic in the surface tangent to that intersection at p is also a geodesic in M .