

On Some Properties of Composition Operators

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1. Introduction. In his thesis [6] Schwartz showed that analytic functions mapping the unit disc into itself induce bounded linear operators (composition operators) on the Hardy space H^2 . He obtained several properties of these operators and in particular obtained a characterization of the compact composition operators, (see all J. V. Ryff [5]). This work has also been extended in several directions. E. Nordgren in [4] describes the spectrum of the composition operators whenever the analytic function inducing the operator is an automorphism of the disc.

Deddens in [2] has also described the spectrum of a composition operator induced by a linear polynomial mapping the disc into the disc. In a more recent work Shapiro and Taylor [7] have obtained conditions on the analytic mapping function in order that the operator be Hilbert-Schmidt or Trace Class. They also have several interesting results concerning when the analytic mapping induces or fails to induce a compact operator. Many of the Shapiro and Taylor results are obtained by placing geometric conditions on the range of the mapping function. In our work we characterize the composition operators which are Fredholm operators. We also obtain a sharp result on when a composition operator has closed range. Our results are stated in terms of the boundary values of the analytic mapping and not in terms of its range. We include a few pertinent examples.

2. Notations and definitions. We let Δ denote the open unit disc in \mathbf{C} and $\Gamma = \{|\lambda| = 1\}$ its boundary. For $1 \leq p < \infty$ the space $H^p(\Delta) = H^p$ consists of the analytic functions f in Δ (with pointwise operations) which satisfy

$$(2.1) \quad \sup_{0 \leq r < 1} \left(\int |f(re^{i\theta})|^p d\theta \right)^{1/p} < \infty.$$

The linear space H^p equipped with the norm (2.1) is called the Hardy space for Δ . If ϕ is an analytic mapping of Δ into Δ then ϕ induces a linear operator C_ϕ on H^p as follows. For $f \in H^p$